The Mathematical Successes and Failures of Students in an Introductory Physics Course

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Introduction

Physics education research is a broad field that covers everything from pedagogical techniques in the classroom to the thought processes of students as they attempt to understand a topic. This thesis focuses on a single aspect of physics education: mathematics. Mathematical capability is paramount to success in any quantitative physics course. Many studies have found a correlation between basic math skills and physics final exam scores. Another study observed that learning gains are correlated with math pretest scores in three of four classes while the pretest scores on the Conceptual Survey in Electricity (which measured physics knowledge) did not correlate with learning gains in any of the four classes. Additionally, mathematical ability correlates positively with success even in a service physics course where the qualitative characteristics of a system are considered more important than numerical values.[1-3]

Despite the importance of math to physics success, there is a massive amount of evidence to suggest that many students are ill-equipped to solve the problems they face in a typical introductory physics course. These problems span a variety of essential math skills, including vector math, trigonometry, and algebra. Many papers have been published regarding student understanding of vectors. One study gave a seven-question diagnostic to students in introductory physics courses, and found that in some classes, over half of the students were unable to add vectors in two dimensions, an essential skill in almost every standard quantitative physics course.[4] Another very large study of vectors identified ten vector concepts that are central to physics and developed a test to identify the main misconceptions for each concept that were causing difficulties for students.[5] Other researchers have looked into physics students' understanding of the trigonometric functions.[6] There is also trigonometry research in the math education community that can yield useful insights into the plight of physics students. For instance, there is evidence that typical students lack the abilities to fully visualize the geometric properties of the trigonometric functions.[7]

This thesis has two goals: to develop a preliminary basis for a longer study documenting the mathematical difficulties of introductory physics students and to identify some of the ideas and abilities that the more successful students use to solve problems. For the most part, this thesis will focus on math problems devoid of any physical context; the problems the students will be asked to solve have no physics concepts embedded in them. This differs from studies which examine how students deal with math while solving physics problems. At the end of this thesis, I will present a set of evidence-based techniques and ideas that might allow students to better solve the math problems that they will encounter in a physics classroom.

Literature Review

The literature mentioned in this thesis is only a small part of the plethora of papers that have been written about physics education. There have been numerous relevant papers written by members of the physics education community, and many of those will be mentioned here. There is also some work done in the math education community that is relevant to this thesis. Often, these papers do not focus on physics, but they do address the same issues: students' struggles trying to solve problems with math. However, the bulk of the literature referenced here is based in physics education. There are many different mathematical methods that students use in a physics course, including graphs, vectors, trigonometry, and algebra.

Reading papers has provided an immense amount of background and helpful ideas that motivated the decisions in the interviews and diagnostics. Between the physics and math education research communities, there are more relevant papers than I could ever hope to read. The focus of these papers varies from high school to upper division undergraduate courses. The most useful (and most plentiful) papers focused on introductory undergraduate courses, but there are still lessons to be learned from papers that are focused on other grade levels.

Graphs

Graphs occur frequently is most introductory physics courses. As an example, students in introductory mechanics are frequently presented with or asked to create graphs of position, velocity, and acceleration vs time. Without a clear understanding about how graphs work, students cannot expect to fully comprehend or appreciate these exercises. While there was no original research done on graphs for this thesis, there is still information to be found in the literature. One study probed students' aptitude at examining the slope and derivative of functions. They found that a relatively high percentage of students (about 85%) could correctly find the slope of a function at several different points and compare them. They also found that when those same students were asked to rank the derivatives of several functions at a point, some of them used the second derivative instead, yielding incorrect answers.[8] This suggests that student difficulties may not be a problem with understanding slope, but rather a failure to connect the idea of a derivative with the slope of a function.

There are many other papers that address graphs in a physics context. These papers can still give insights into students' understanding of graphs. Several very thorough studies of graphs in physics found many common difficulties for students, including thinking of the graph as a picture of the situation, confusing the value of the function with its slope, confusing different variables such as position and velocity.[9-10] Another study found that many students struggled with negative velocity values on a graph, thinking that they should be positive (like speed) instead of negative (because velocity has a direction).[11] Finally, researchers found that when students are graphing complex, piecewise functions, they benefit from graphing each part separately before graphing the entire thing.[12] These are just a few of the findings that are present in the literature on graphing, but that is not the only type of math physics students struggle with.

Trigonometry is a topic that is not discussed frequently in the physics education literature. One study looked at students' understanding of the trigonometric functions in a non-major introductory physics course and found that many students lack a conceptual understanding of nuances of the trigonometric functions. For instance, they found that many students could not explain why the term $2\pi f$ is used to set the frequency of a cosine function. Interestingly, they found that students performed relatively well on questions about right triangles (90% correct), which is quite different than the results found on the diagnostics used in our study (see Findings section).[6] There is also mathematics education research that is relevant to this thesis. One study at the high school level found that students have significant struggles with more complex trigonometry problems. As an example, only 9% of students correctly solved for x when $sin30° = x$.[13]

Much like trigonometry, vectors are a mathematical tool that most introductory physics students deal with on an almost daily basis. One essential study identified 10 concepts in vector math that physics students should know, including the direction and magnitude of a vector, components, and vector addition and subtraction.[5] However, previous research and work done for this thesis suggest that many students lack the knowledge to perform even basic vector operations. Studies have found that some students think of vector direction as an approximate instead of exact quantity, so they would consider two vectors to have the same direction if they both pointed up and to the right for example.[5] Another study found that a distressingly small percentage of students (as low as 22% in one algebra-based class) were able to correctly add vectors in two dimensions.[4]

Many researchers have examined how students add vectors. A small study found that tip-to-tail was the most popular method for vector addition by students, but that components were also a popular way to add vectors.[14] A separate study found that while arrow (tip-to-tail) and component methods had similar results on vector addition, students who used components had a significantly higher success rate on vector subtraction.[15]

Graphs, trigonometry, and vectors are all examples of mathematical tools that students struggle with. But students can also struggle with problems if they are presented in different ways. Math problems can be given with numbers, symbolic variables, or other numeric representations as the coefficients in a problem. Several papers have found that success rates can be as much as 50% higher on numeric problems, but that this discrepancy isn't found across all student levels. Instead, the poorest students had a very large difference between symbolic and numeric problems, while the highest-performing students scored roughly the same on both types of questions.[16-17] It remains to be seen whether these students score better on the symbolic problems because they have a better understanding of mathematics, or if instead a nuanced understanding of mathematics that allows them to score better on the symbolic problems also helps them outperform their classmates on numeric problems as well.

The fact that so many students struggle with such simple problems has significant implications for the physics classroom. According to cognitive load theory, when a student struggles to solve one of the preliminary steps of a problem (like finding the length of the side of a triangle), there are fewer cognitive resources left to solve the rest of the problem.[18] So until students become proficient in the necessary mathematics, they will not be able to fully commit their efforts to the physics.

However, there are ways to address these issues. Multiple studies have found that a shift in instructional style and supplementary tutorials can significantly improve student performance on math problems involving trigonometry and vectors.[7,14]

Procedure

The primary method for research involves giving short diagnostics to a large number of students. These diagnostics were mainly given in the spring and fall semesters of 2016 at ASU to four different physics courses, PHY 111, 112, 121, and 131. PHY 111 and 112 cover mechanics and electricity & magnetism, respectively. These classes are algebra-based and meant for non-engineering students. To enroll in PHY 111, students must have taken Precalculus, or be concurrently enrolled in Brief Calculus. PHY 121 and 131 also cover mechanics and electricity & magnetism, but they are meant to be calculus-based and directed toward engineering students. Calculus 1 is a prerequisite and Calculus 2 is a corequisite for PHY 121. As a note, the classes for physics majors, PHY 150 and 151, were avoided for the study.

Before each semester started, professors were asked to use the first recitation or lab day to give the exams. In the spring, all the diagnostics were given on the Polytechnic campus. There were 257 students that took the exam. The class breakdown is as follows: 72 in PHY 111, 52 in PHY 112, 104 in PHY 121, and 29 in PHY 131. In the fall, some diagnostics were given on the Polytechnic campus, and some on the Tempe campus. The PHY 111, 112, and 121 students were all on the Polytechnic campus, and the 131 students were on the Tempe campus. A total of 679 students took the fall diagnostic; the breakdown is: 94 in PHY 111, 54 in PHY 112, 98 in PHY 121, and 433 in PHY 131. The large proportion of students in PHY 131 is simply because one of the professors that agreed to give the diagnostic had multiple 200 student lectures; this was not optimal for the study in any way. It’s also important to note that the spring and fall diagnostics are not identical, as some (but not all) of the questions were modified, eliminated or added for the second diagnostic. There were also two versions of the fall diagnostic that had small differences. Students were usually given about half an hour to complete the diagnostic at the end of the first recitation or lab. According to the professors that administered the diagnostics, almost all students finished in that time.

The diagnostics are the source of the bulk of the data for this thesis, with over 10,000 data points (over 10 questions per diagnostic, with almost 1,000 diagnostics). They are effective for getting data on a large number of students. However, they give a limited amount of information per student. It is often difficult to determine a student's thought process from the work on a multiple choice question. To dig deeper into student's understanding, interviews are necessary.

The one-on-one interviews provide a much more thorough and in-depth examination of student thought processes. Unlike a diagnostic, which requires deducing the students' thoughts from the work left behind, an interview allows the researcher to explicitly ask the student what they were thinking when solving that problem. The obvious downside of interviews is the time required for each interview. The interviews are typically broken into two parts, each lasting roughly 20-30 minutes. First, the students work through the problems (the same questions as the diagnostics). After that, the student is recorded as they explain how they solved each problem. The researcher can ask questions whenever necessary, but it is typically most effective to let the student do the bulk of the talking. It is important not to plant ideas in the students' minds that are not their own, or to remind them of things they had forgotten.

For this thesis, I conducted 12 interviews, all in the spring semester. There was at least one student from each of the four physics courses (111, 112, 121, 131), but not all of the interviewees took the diagnostics. 8 of the students were on the Tempe campus and did not take the diagnostic at the beginning of the semester. To get more interview participants, some students on the Tempe campus were invited to participate in the interviews. It is also important to note that many of the interviews were done late in the semester, with several taking place in April. This means that the interview scores are probably inflated relative to the diagnostics, as students have had up to two months learning and developing these mathematical techniques in class before being interviewed. Because of this, a direct comparison between the interviews and the diagnostics should only be made with extreme care.

Findings

Before presenting my results, it is important to note that these are preliminary results. The findings below are not definitive; rather, they should be reexamined and amended as more data (particularly from interviews) is collected. Whenever possible, I will try to recommend the next steps to dig deeper into a particular topic. First, I will go over some of the topics that students struggle with, and then I will explain the methods uncovered in this study that students use to effectively solve problems.

Student Difficulties

Trigonometry:

I did a significant amount of original research regarding trigonometry; about 1/3 of the questions on the diagnostics and interviews involved trigonometry. Over the course of this thesis, it has become apparent that trigonometry is an area that causes problems for many students. For example, the following problem was given on every diagnostic:

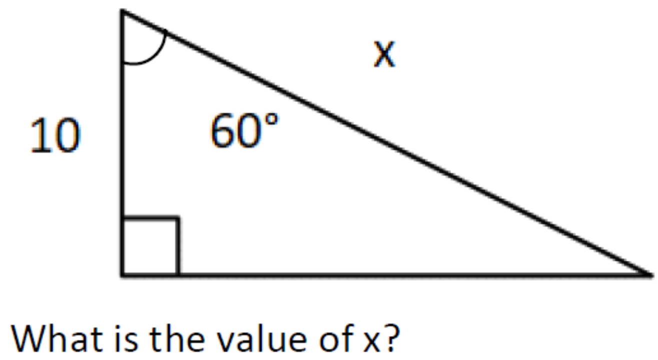


Figure 1: Diagnostic question that requires the use of a trigonometric function; Correct Answer: 20

This problem is much simpler than the typical problem in physics, but the success rate for this problem was typically only about 50%. Some class responses from the fall diagnostics are below:

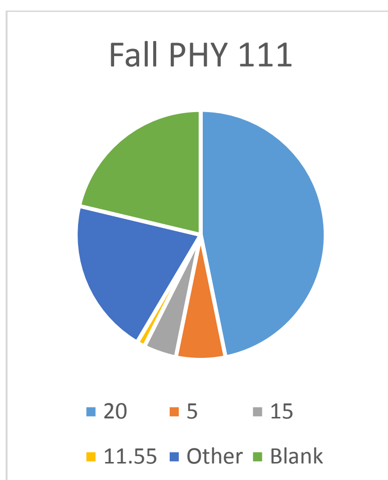


Figure 2: Fall PHY 111 responses to the question in Figure 1; N=94

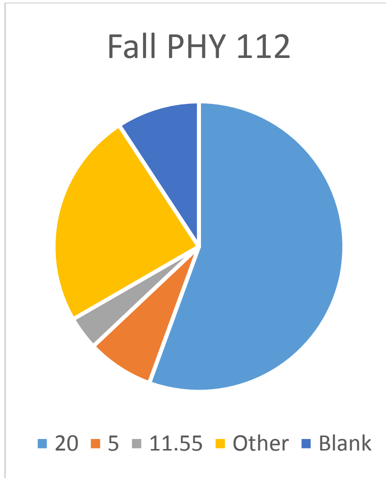


Figure 3: Fall PHY 112 responses to the question in Figure 1; N=54

(Notice that the percentage of correct answers increases from the first to the second semester courses.) This problem is the easiest version of a trigonometry problem, and over half the students in the first semester courses failed to answer it correctly. The most common wrong error was 5, and was caused by an algebra error or incorrectly using a trigonometric function, such as the example below:

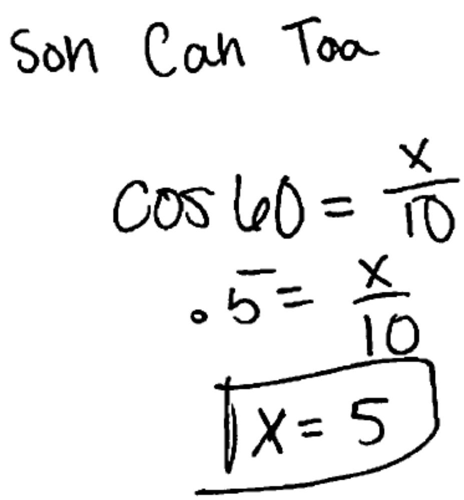


Figure 4: Example of incorrect student work for question in Figure 1

Despite writing down the well-known mnemonic SOH CAH TOA, the student above still wrote the incorrect relationship, leading to a wrong answer.

Students have similar difficulties with the inverse trigonometric relationships. A similar problem is shown below, but this time the two sides are given and the student is asked to find the angle.

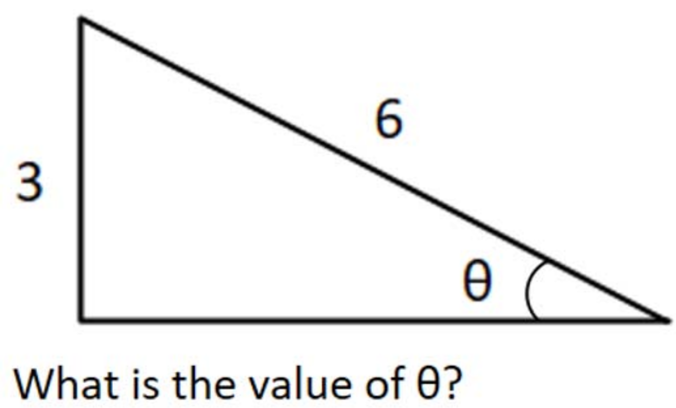


Figure 5: Diagnostic question requiring an inverse trigonometric function; Correct Answer: 30°

The following figures show responses to this question:

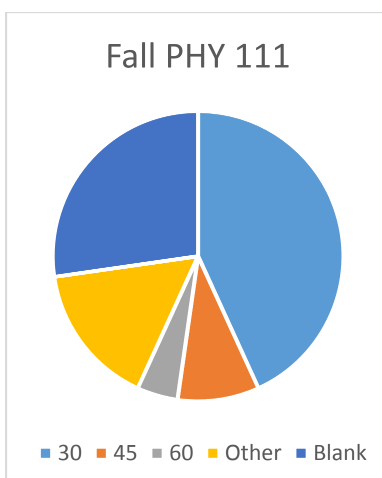


Figure 6: Fall PHY 111 responses to the question in Figure 5; N=94

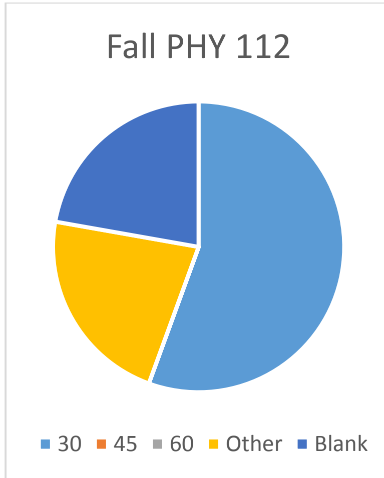


Figure 7: Fall PHY 112 responses to the question in Figure 5; N=54

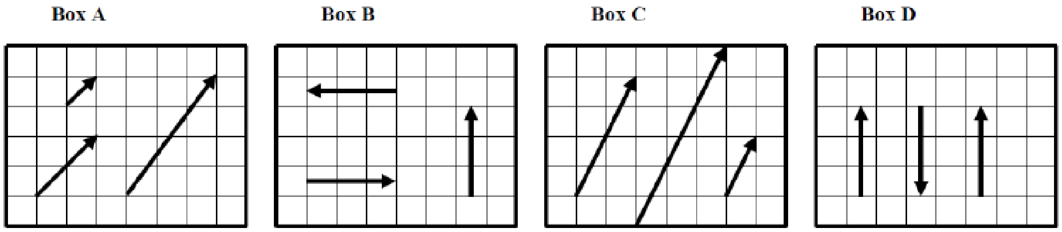
Note that the answer is in degrees; however, a significant portion of students gave only the number, with nothing to indicate that response was in degrees.

Figures 6 and 7 correspond to the same student populations as Figures 2 and 3, respectively. Comparing the two charts shows that there is almost no difference in the success rate for the problems shown in Figures 1 and 5. This suggests that most students do not have much difficulty taking the inverse of a trigonometric function, so it seems likely that improving student performance with the trigonometric functions will automatically increase their performance with the corresponding inverse functions.

Vectors

There were also questions about vectors in the diagnostics and interviews. In the spring, students were asked to compare vector directions and add vectors in one dimension. In the fall, one question on the diagnostic involved 2-D vector addition. Many of the student responses confirmed things that have already been reported by other studies. For instance, Barniol and Zavalla[5] found that many students thought that vectors that pointed in the same general direction had the same direction. This was verified during the interviews, where 25% of the students said the vectors in Box A of Figure 8 had the same direction. C is the correct answer.

7. In the four boxes below are collections of vectors on top of equally spaced grid lines. Choose the answer from the list below that most correctly describes the comparative **directions** of the vectors within each box.



Possible answers. Select the best one.

A. Box **A** has all vectors with the same direction
B. Box **B** has all vectors with the same direction
C. Box **C** has all vectors with the same direction
D. Box **D** has all vectors with the same direction
E. Both boxes **A** and **C** have vectors that all have the same direction
F. Both boxes **A** and **D** have vectors that all have the same direction
G. Both boxes **C** and **D** have vectors that all have the same direction
H. The boxes, **A**, **C**, and **D** have vectors that all have the same direction
I. None of the above boxes have vectors with the same direction

Figure 8: Vector direction question on spring diagnostics and interviews; Correct Answer: C

Question taken from Vector Concept Quiz[19]

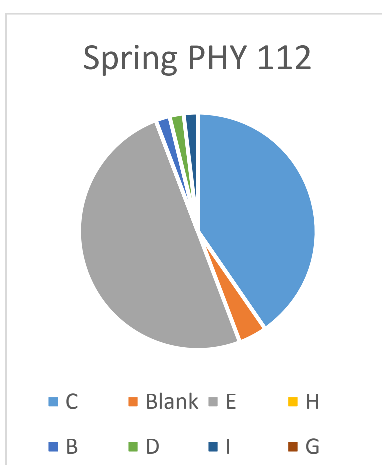


Figure 9: Spring PHY 112 responses to the question in Figure 8; N=52

Figure 9 shows the responses on the vector direction problem in Figure 8. Roughly half of the students selected E, which included all the vectors in Figure 8 being in the same direction.

Questions on the diagnostic also confirmed that many students struggle with vector addition. There were three different vector addition questions asked: two vectors in one dimension (Figure 10 below), two vectors in two dimensions (Figure 13 below), and three vectors in two dimensions (Figure 17 below).

The students asked to add two vectors in one dimension had success rates in the 50-80% range. The responses of the PHY 111 and 121 students are shown below. (The correct response was C.)

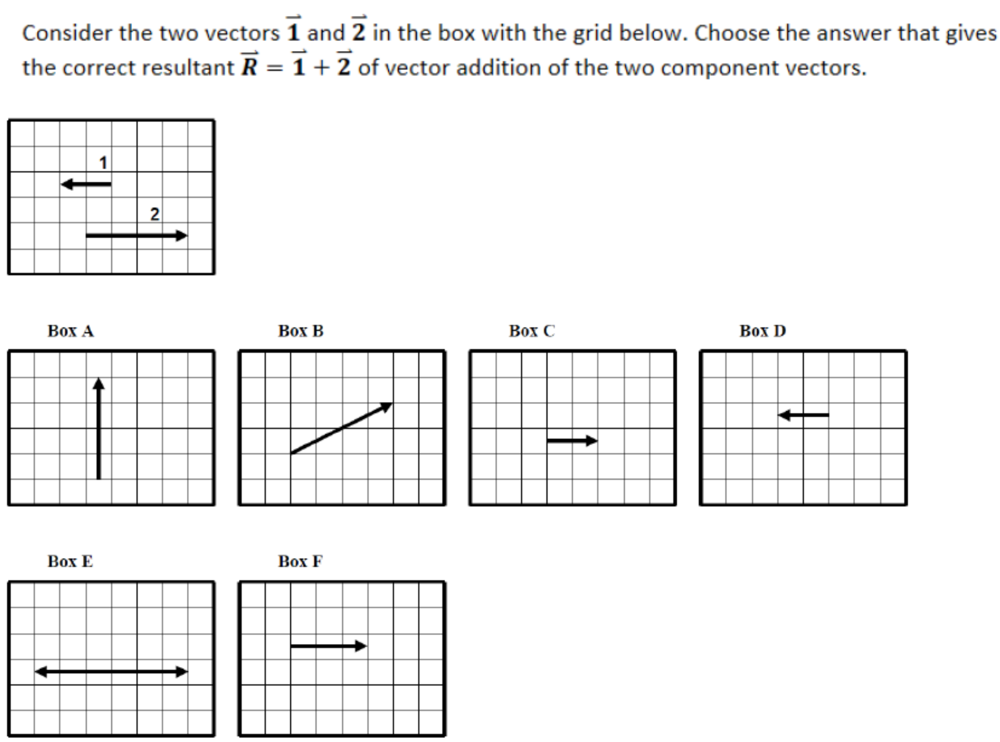


Figure 10: 1-D vector addition question on the spring diagnostics; Correct Answer: C

Question taken from Vector Concept Quiz[19]

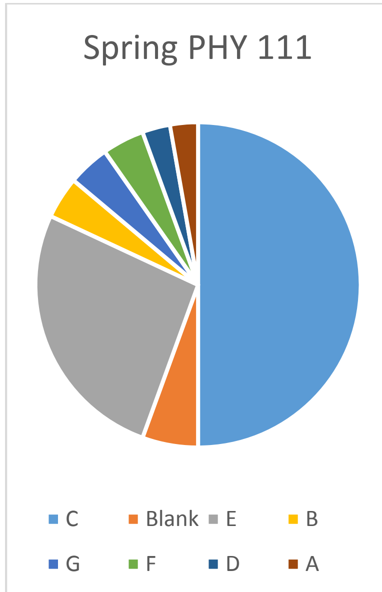


Figure 11: Spring PHY 111 responses to the question in Figure 10; N=72

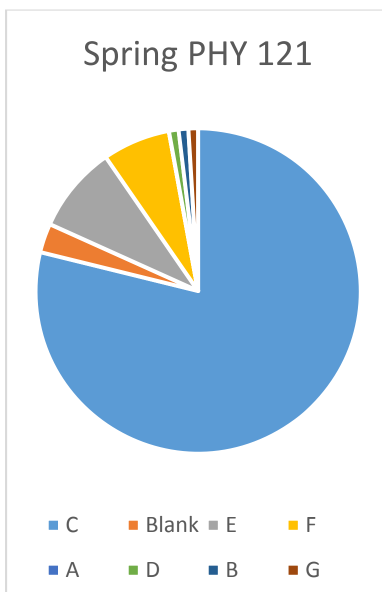
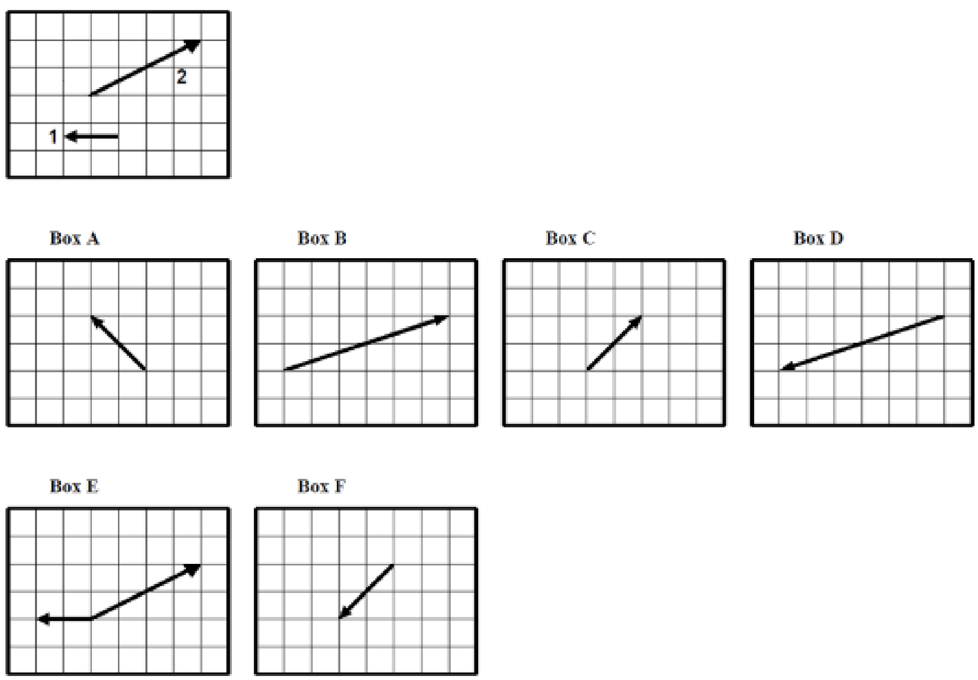


Figure 12: Spring PHY 112 responses to the question in Figure 10; N=52

The most common wrong answer, E, isn't even a vector! This sugests that a significant portion of students (~25% in PHY 111) don't even have a correct idea of what a vector is. Additionally, when faced with a two-dimensional problem like the questions below, student responses get even worse.

5) Consider the two addition of the two cctors $\vec{\mathbf{1}}$ and $\vec{\mathbf{2}}$ in the box with the grid below. Choose the answer that gives the correct resultant $\vec{\boldsymbol{R}} = \vec{\mathbf{1}} + \vec{\mathbf{2}}$ of vector omponent vectors.



Possible answers. Select the best one.

A) Box **A** has the vector that is the correct resultant $\vec{\boldsymbol{R}}$
B) Box **B** has the vector that is the correct resultant $\vec{\boldsymbol{R}}$
C) Box **C** has the vector that is the correct resultant $\vec{\boldsymbol{R}}$
D) Box **D** has the vector that is the correct resultant $\vec{\boldsymbol{R}}$
E) Box **E** has the vector that is the correct resultant $\vec{\boldsymbol{R}}$
F) Box **F** has the vector that is the correct resultant $\vec{\boldsymbol{R}}$

Figure 13: 2-D vector addition with two vectors, included on one version of the fall diagnostic; Correct Answer: C

There were no PHY 121 classes that took this version of the diagnostic, but the results for the other three classes are below (C is the correct answer):

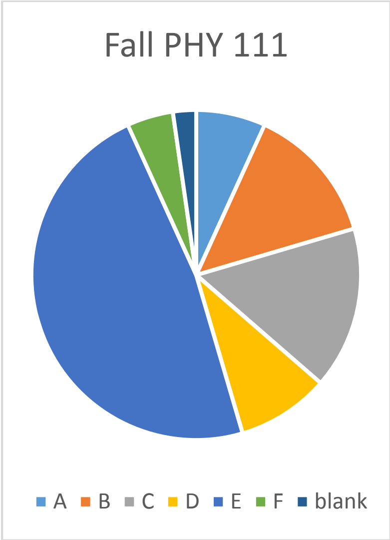


Figure 14: Fall PHY 111 responses to the question in Figure 13; N=94

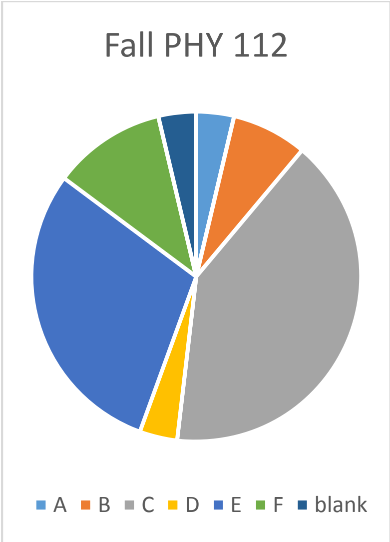


Figure 15: Fall PHY 112 responses to the question in Figure 13; N=54

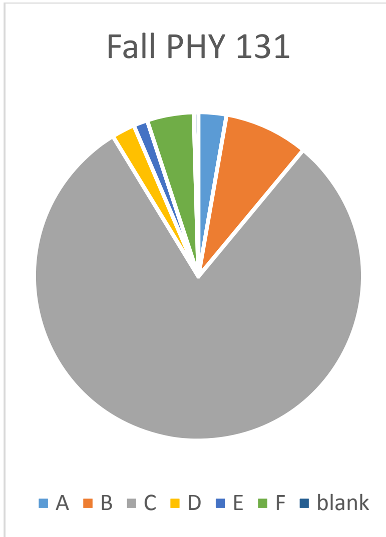
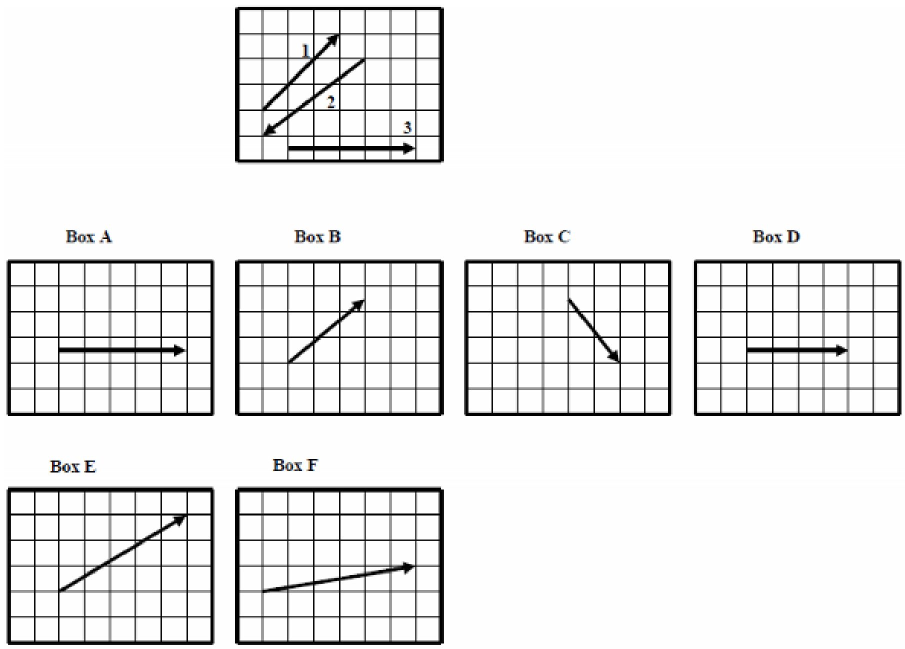


Figure 16: Fall PHY 131 responses to the question in Figure 13; N=433

Only about one sixth of PHY 111 students answered this question correctly. Even in second semester calculus based physics, ~25% of students answered this question incorrectly.

Less than a quarter of students in PHY 111 were able to add two very simple vectors in two dimensions. However, when a third vector is added, the scores go down further.

5) In the figure below are given three vectors $\vec{\mathbf{1}}$, $\vec{\mathbf{2}}$, and $\vec{\mathbf{3}}$. There exists a resultant sum, $\vec{\boldsymbol{R}}$, of the vector addition of the three component vectors (i.e., $\vec{\boldsymbol{R}} = \vec{1} + \vec{2} + \vec{3}$). Find the best choice out of the given boxes below where the vector shown most closely resembles the correct resultant of the vector addition of the three component vectors.



Possible answers. Select the best one.

A) Box **A** is the best choice for the resultant $\vec{\boldsymbol{R}}$
B) Box **B** is the best choice for the resultant $\vec{\boldsymbol{R}}$
C) Box **C** is the best choice for the resultant $\vec{\boldsymbol{R}}$
D) Box **D** is the best choice for the resultant $\vec{\boldsymbol{R}}$
E) Box **E** is the best choice for the resultant $\vec{\boldsymbol{R}}$
F) Box **F** is the best choice for the resultant $\vec{\boldsymbol{R}}$

Figure 17: 2-D vector addition with three vectors on one version of the fall diagnostic; Correct Answer: D – Question taken from Vector Concept Quiz[19]

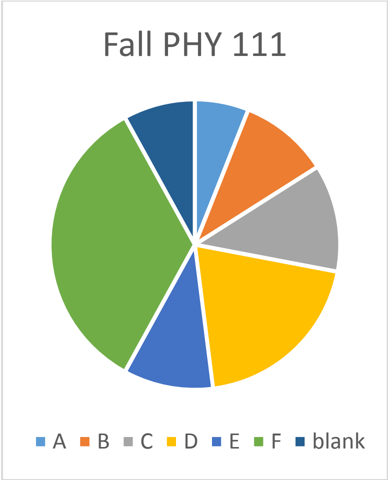


Figure 18: Fall PHY 111 responses to the question in Figure 17; N=94

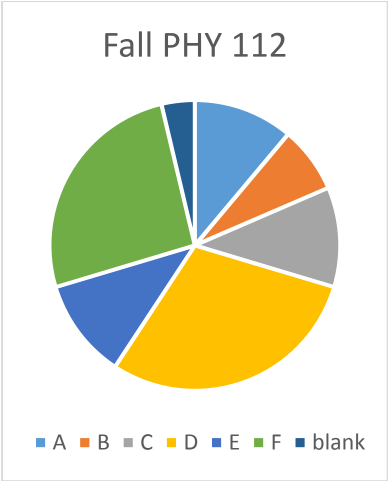


Figure 19: Fall PHY 112 responses to the question in Figure 17; N=54

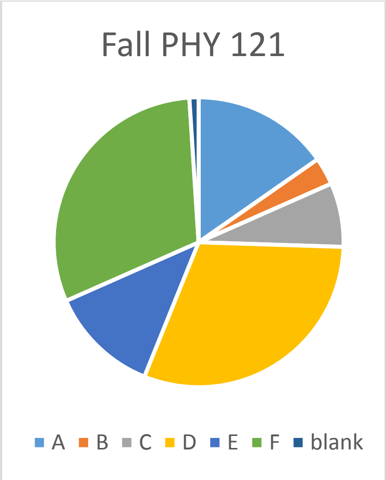


Figure 20: Fall PHY 121 responses to the question in Figure 17; N=98

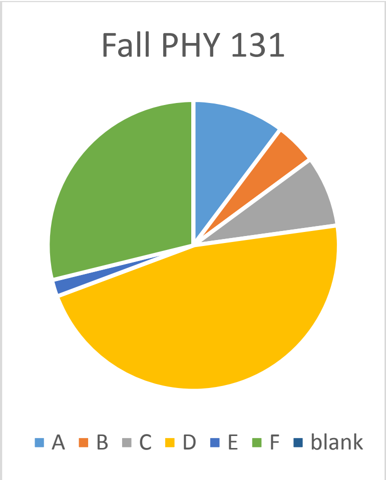


Figure 21: Fall PHY 131 responses to the question in Figure 17; N=433

Notice that as the problems increase in complexity, more and more students get incorrect answers. So perhaps it is the multi-step nature of these problems that is creating some of the confusion rather than a fundamental lack of understanding about vectors. However, the prevalence of answers that aren't even vectors suggests that something is seriously wrong with the basic knowledge of a significant portion of students.

Algebra

There is an even more basic branch of mathematics in which students frequently make mistakes: algebra. Most problems, such as the trigonometry problems already shown, require at least some algebra to get to the final answer. Other questions only involve algebra. Both types of problems were examined on the diagnostics, and both provided many examples of students making small mistakes on algebra. For example, in the spring, students were asked to solve the following problem:

10. Solve for *y* as a function of *x* from the following equations:

$5x - 3t = 2$

$y = 3(3t + 2)$

Figure 22: Problem on spring diagnostic involving parametric equations; Correct Answer: y=15x

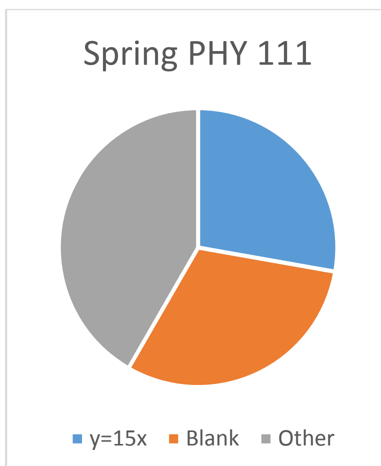


Figure 23: Spring PHY 111 responses to the question on Figure 22; N=72

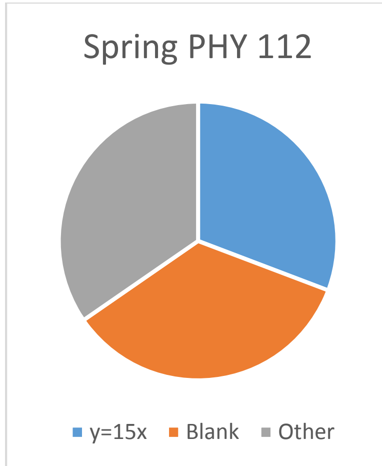


Figure 24: Spring PHY 112 responses to the question on Figure 22; N=52

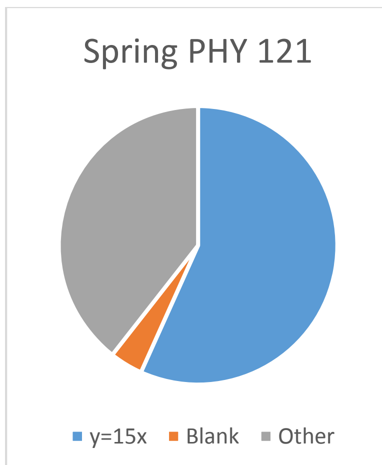


Figure 25: Spring PHY 121 responses to the question on Figure 22; N=104

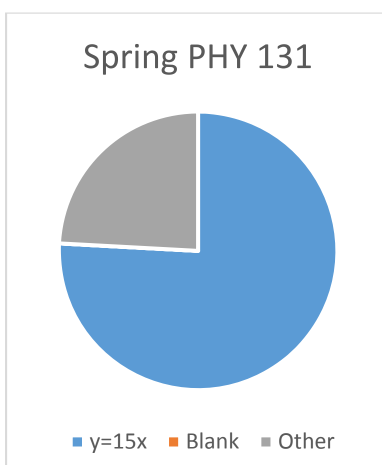


Figure 26: Spring PHY 131 responses to the question on Figure 22; N=29

To learn more about why so many students were getting this problem wrong, the students' work was closely analyzed to identify common errors. Out of the 96 errors, I was able to identify the source of the incorrect answer on 61 of them, and of those 61, 45 made algebra errors that led to incorrect answers.

Students had more success on simpler problems. For example, the fall diagnostics had the following question:

6) What is the value of x if $x = p(p + q) + 4$, $p = -2$, and $q = 5$?

A) -2
B) -3
C) 4
D) -6
E) 18

Figure 27: Simple algebra problem on the fall diagnostics; Correct Answer: A

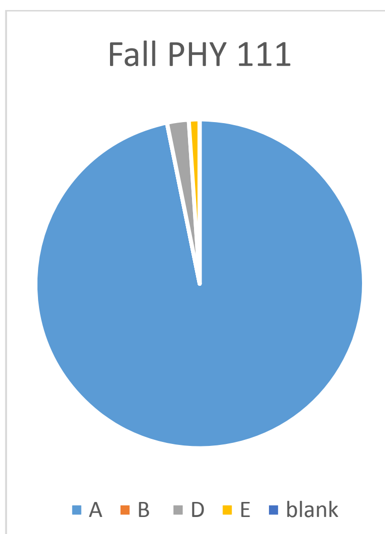


Figure 28: Fall PHY 111 responses to the question in Figure 27; N=94

As shown in Figure 28, student response to this problem was far better than the problem in Figure 22. This confirms that the vast majority of students know how to perform basic algebra operations including: substitution, addition, multiplication, and order of operations. However, they frequently make algebra mistakes when solving more complicated problems, such as those found in Figure 22. The only additional skill needed to solve Figure 22 is operating on both sides of an equation, so perhaps this is the fundamental skill that many students are lacking. Alternatively, it is conceivable that the complexity of the problem overwhelms the students, causing them to "freeze up."

Fractions

There are other researchers investigating students' mathematical capabilities at ASU. As part of a separate study, Dr. Jose Menendez[20] gave a multiple choice test to PHY 121 students to test their math capabilities. Compare the results from the following two questions.

Calculate $\frac{2}{7}+\frac{1}{21}$	
1	$\frac{1}{3}$
2	$\frac{3}{28}$
3	$\frac{2}{147}$
4	$\frac{5}{21}$
5	$\frac{8}{21}$

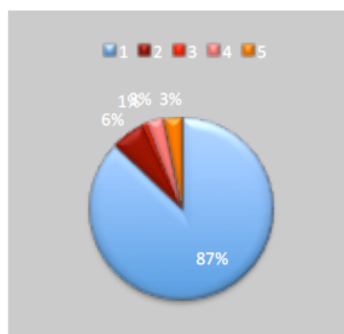


Figure 29: Student responses to a fraction addition problem; Correct Answer: 1/3

Erica bought $3\frac{1}{2}$ yards of fabric. If she uses $\frac{2}{3}$ of the fabric to make a curtain, how much will she have left?	
1	$1\frac{1}{6}$ yard
2	$\frac{1}{6}$ yard
3	$\frac{1}{3}$ yard
4	$2\frac{1}{3}$ yard
5	$2\frac{5}{6}$ yard

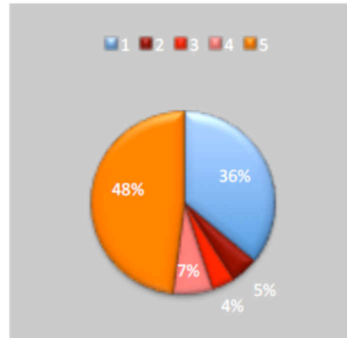


Figure 30: Student responses to a fraction multiplication problem; Correct Answer: $1\frac{1}{6}$ yards

Both of the above questions involve computation with two fractions, but the success rate for the first is more than double the second. There are two possible reasons for this. First, it is possible that students are just more comfortable adding fractions than multiplying them. The second possibility is that students have a difficult time getting from the word problem to the equation $3\frac{1}{2} * \frac{1}{3} = \frac{7}{2} * \frac{1}{3} = x$. If this is the case, it is not actually the mathematical content of the problem that is difficult for students, it is the way it is asked. This seems more likely, but it is impossible to assert with any confidence without additional research.

Careless Errors

There is another problem for many students that is often disguised as an algebra error. During the interviews, when working through the questions the first time, students would occasionally make mistakes, often algebra errors. However, quite frequently when explaining their work, they noticed the mistakes and corrected them. Over 12 interviews, there were 35 questions with incorrect answers. Of those 35, 12 contained errors that the students recognized and fixed. So only about two thirds of the errors were fundamental misconceptions or problems; the rest were the result of going too quickly, skipping steps, etc. These errors included accidentally multiplying instead of dividing, using the wrong trigonometric function, and typing an expression into the calculator incorrectly.

The prevelence of careless errors (which do not represent a fundamental misunderstanding of a mathematical concept) means that all the diagnostic results should be taken with a healthy amount of skepticism. For instance, when examining the cause for the incorrect answers in solving the parametric equations in Figure 22, it was found that 45 of the 61 diagnosable incorrect answers were algebra errors. But it seems plausible that a significant number of those errors were actual careless errors, and not an incorrect understanding of algebra. It is now especially important to be watching for careless errors in future interviews.

Symbolic Notation

There is also some evidence to suggest that students are more successful solving problems that use ordinary numbers (i.e. 5, 3/4, 24.7) rather than constants (a, b, c) or functions (cos(20), sin(40)). During the interviews, the students were asked to solve the following problem:

$$x \cos 20° = y \cos 70°$$

$$x \cos 70° + y \cos 20° = 10$$

$$x = ?$$

$$y = ?$$

Figure 31: System of equations for students to solve on the diagnostics and interviews; Correct Answer: x=3.4, y=9.4

Multiple students claimed that the problem was more difficult because the coefficients of the variables were not ordinary numbers. For example, one student got confused because $cos20$ and $cos70$ looked similar. To further investigate this, the fall diagnostics had two versions of this question, one with cosines and one with constants. However, there was no statistically significant difference between the results of two problems, indicating that students may find symbolic and functional coefficients to be of equal difficulty.

Student Strategies for Solving Problems

Poor strategies

SOH CAH TOA

Now, before explaining some concepts that would seem to most benefit physics students, it may also be helpful to dispel (or at least cast doubt on) several popular problem-solving strategies that preliminary results suggests may not be as effective as many people believe. In trigonometry, the (in)famous SOH CAH TOA is an extremely ingrained part of most students' understanding; it is a mnemonic device that helps many students remember which trigonometric relationship relates certain sides of a triangle (so as an example, sine [S] corresponds to the opposite [O] divided by the hypotenuse [H]).

Surprisingly, the diagnostics provided some evidence that would suggest that SOH CAH TOA is not a magic bullet, as many students and teachers believe, and in fact might even be a crutch that gives students a false sense of confidence regarding trigonometry.

To examine the effectiveness of SOH CAH TOA, one of the trigonometry problems on the diagnostic was examined to see if the expression SOH CAH TOA appears in the students' work.

The following is an example of a student who obviously knows and considered SOH CAH TOA but still got the problem wrong.

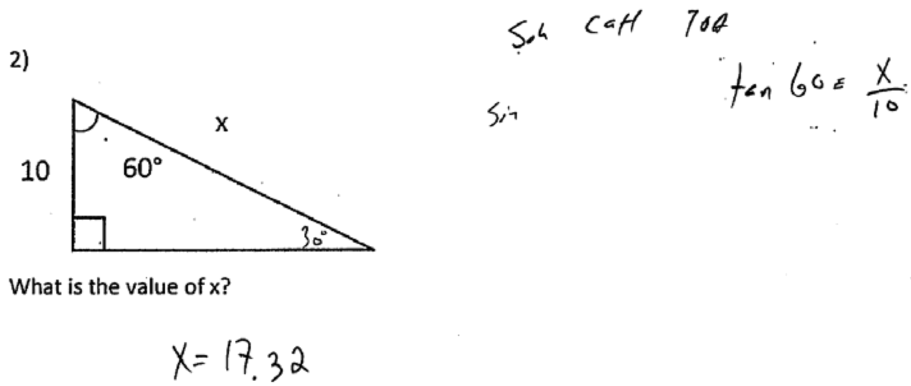


Figure 32: Example of student work with SOH CAH TOA and the wrong answer

Then a chi squared test was done to test the statistical significance between writing SOH CAH TOA and getting the answer correct. For the fall semester, data was collected separately for each physics class. The results are below:

PHY 111	Number Correct	Number Incorrect		χ^2	p value
With SOH CAH TOA	11	7		2.6245	0.10
Without SOH CAH TOA	30	45			
PHY 112	Correct	Incorrect			
With SOH CAH TOA	10	3		3.166	0.08
Without SOH CAH TOA	20	21			
PHY 121	Correct	Incorrect			
With SOH CAH TOA	5	12		1.5182	0.22
Without SOH CAH TOA	37	44			
PHY 131 (Tempe)	Correct	Incorrect			
With SOH CAH TOA	32	10		5.4778	0.02
Without SOH CAH TOA	347	44			

Table 33: Results of chi squared test, showing the range of the p value, demonstrating statistical significance

The p value is in the range shown on the far right column. Notice that the only class with a p value less than 0.05 (the standard for statistical significance) is PHY 131, and it actually signifies a statistically significant correlation between writing SOH CAH TOA and getting the answer incorrect! At first glance, this result seems to fly in the face of common belief. However, there are a few things to keep in mind when interpreting these results. First, this only counts the students that wrote SOH CAH TOA. It is not only possible but extremely likely that many students thought of SOH CAH TOA and used it to solve the problem without explicitly writing it down on the paper. Second, the students in PHY 131 have already taken Calculus 1 and PHY 121, both of which use trigonometry, so it is conceivable that these students no longer need SOH CAH TOA to remember the trigonometric functions. However, this does not imply that SOH CAH TOA was not useful to them in developing their understanding of the trigonometric functions. Further research is almost certainly needed to make any credible claim for the effects of SOH CAH TOA on student performance.

Vector representation: tip-to-tail

Returning to vector addition, one of the interview questions was the following one dimensional vector addition problem.

Consider the two vectors **1** and **2** in the box with the grid below. Choose the answer that gives the correct resultant $\vec{R} = \vec{1} + \vec{2}$ of vector addition of the two component vectors.

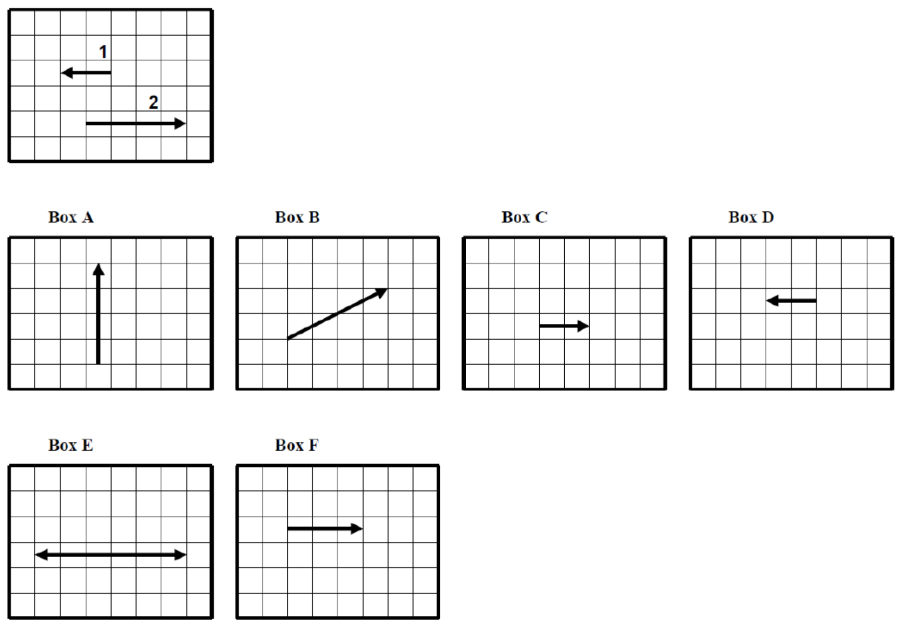


Figure 34: 1-D vector addition problem asked during interviews; Correct Answer: C

Question from Vector Concept Quiz (by David Meltzer)

5 of the 12 students used some sort of tip-to-tail method to solve this problem. One of those five was still uncertain even when helped through the problem. Unfortunately, this isn't the best problem to test tip-to-tail effectiveness on because the vectors are in one dimension, but other studies have also looked at vector addition. However, tip-to-tail may not be the most effective way to add vectors; research suggests that there is a more effective way to add vectors.

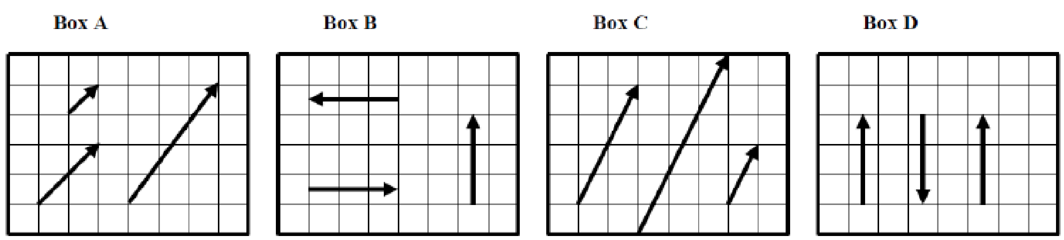
Successful Strategies

Vector representation: components

The other technique the interview students used when answering vector questions was vector component analysis. During interviews, 6 students used component-like reasoning to get the correct answer. Together with the results of a study done by Heckler and Scaife[15], this suggests a possible advantage in using component analysis.

The use of vector components may also be very useful when considering vector direction. The interviewees also answered this question:

7. In the four boxes below are collections of vectors on top of equally spaced grid lines. Choose the answer from the list below that most correctly describes the comparative **directions** of the vectors within each box.



Possible answers. Select the best one.

A. Box **A** has all vectors with the same direction
B. Box **B** has all vectors with the same direction
C. Box **C** has all vectors with the same direction
D. Box **D** has all vectors with the same direction
E. Both boxes **A** and **C** have vectors that all have the same direction
F. Both boxes **A** and **D** have vectors that all have the same direction
G. Both boxes **C** and **D** have vectors that all have the same direction
H. The boxes, **A**, **C**, and **D** have vectors that all have the same direction
I. None of the above boxes have vectors with the same direction

Figure 35: Vector direction problem given on interviews; Correct Answer: C

Question from Vector Concept Quiz (by David Meltzer)

7 of the 12 students looked at the components of the vectors or calculated the slope to see that the vectors in Box A did not all have the same direction, and all of them got the answer correct. Only two students managed to get the correct answer without referencing the components or slope of the vectors. This suggests that using vector components instead of a tip-to-tail representation

may provide an advantage when solving problems involving vector subtraction and direction. However, additional research (particularly interviews) will be necessary to confirm these suspicions.

Practice with the trigonometric functions

I have already suggested that, when solving trigonometry problems, SOH CAH TOA may not be as effective as many people (both teachers and students) think it is. Instead, evidence suggests that it may be more important for students to have lots of practice using the trigonometric functions than that students have SOH CAH TOA memorized. Consider the PHY 131 students:

PHY 131	Correct	Incorrect		χ^2	p value
With SOH CAH TOA	32	10		5.4778	0.02
Without SOH CAH TOA	347	44			

Table 36: Results of chi squared test, showing the range of the p value, demonstrating statistical significance. Only for PHY 131 Tempe students in Fall

It is safe to assume that most of these students have a lot of practice with the trigonometric functions. In fact, there is actually a statistically significant correlation between writing down SOH CAH TOA and getting an incorrect answer. This seems to suggest that as students get more practice with the trigonometric functions, they become more capable at solving trigonometry problems easily, without needing to write out SOH CAH TOA each time. This is backed up by the interviews. Some students could write out SOH CAH TOA and then with some effort solve the problem. However, the more successful students had enough experience with these functions that they were able to instantly write down the equations with no difficulty. This level of aptitude can only be achieved with a significant degree of practice. This result makes sense when you consider cognitive load theory.[18] If students have to spend a lot of cognitive resources figuring out which trigonometric function to use, they have less resources to finish solving the problem. Therefore, when confronted with students who are struggling in trigonometry, it may be more beneficial to have them work through lots of simple practice problems instead of explaining SOH CAH TOA to them again. This way, they are able to set up the problem effortlessly simply by force of habit, instead of struggling to use SOH CAH TOA to begin the problem.

Rechecking work

Perhaps the most surprising result of this research yet is that careless errors may be causing a significant percentage of all incorrect answers by students. In the interviews, a third of all errors were identified and fixed by the students themselves, so it follows that students can have more success by simply checking their work. A diagnostic was administered in one class over the summer semester (there will be additional comments on the summer diagnostic later) that asked students if they checked their work. However, there were very few responses to that question, so it seems that interviews may be the only way to ensure that students check their work and test for careless errors.

Symbolic notation

There are also a few successful tactics that students use to solve systems of equations. It was mentioned before that students struggle more with equations that have symbolic constants (a, b, c) or functions (cos20, sin70) instead of simpler numbers. It follows that students should benefit from seeing and understanding how symbolic constants can be manipulated as easily as numeric constants. If they can learn to treat symbolic variables with the same ease as numeric values, their success rates should increase.

Algebraic "toolbox"

Additionally, there was a common issue with the problem shown below.

8)
$a \cdot \mathrm{x} = \mathrm{b} \cdot \mathrm{y}$
$b \cdot \mathrm{x} + \mathrm{a} \cdot \mathrm{y} = c$

a, b, and c are constants.

A) What is the value of x in terms of a, b, c, and y?
What is the value of y in terms of a, b, c, and x?

B) What is the value of x in terms of a, b, and c, NOT in terms of y?
(Your answer should not have y in it.)
What is the value of y in terms of a, b, and c, NOT in terms of x?
(Your answer should not have x in it.)

Show all your steps.

Figure 37: System of equations for students to solve on the diagnostics; Correct Answer:

$$x = \frac{by}{a} = \frac{c}{b+\frac{a^2}{b}}, y = \frac{ax}{b} = \frac{c}{a+\frac{b^2}{a}}$$

Note that the first part of the problem requires solving for x in terms of y or vice versa before making the appropriate substitution. The figures below show the responses for each of the classes in the fall semester.

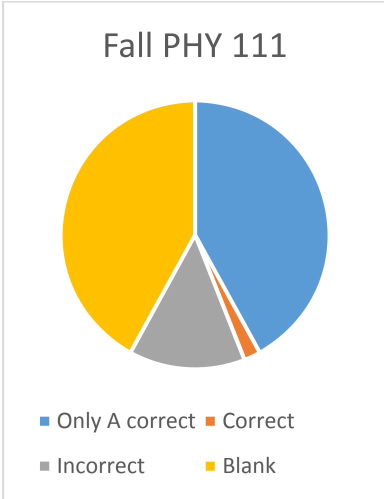


Figure 38: Fall PHY 111 responses to the question in Figure 37; N=94

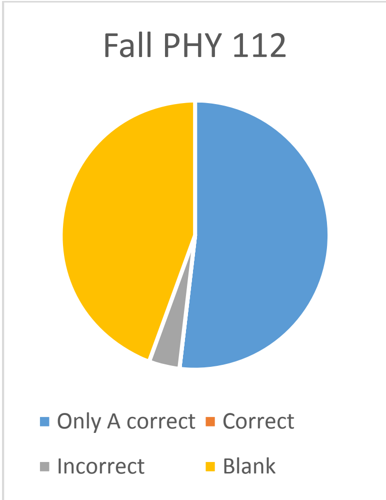


Figure 39: Fall PHY 112 responses to the question in Figure 37; N=54

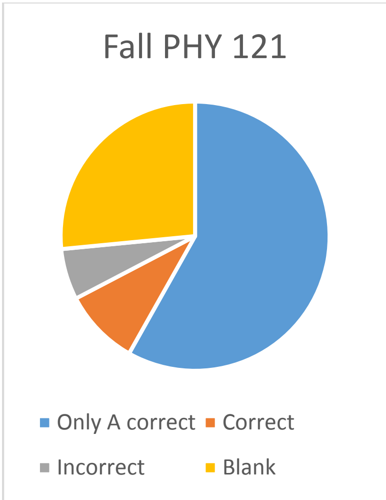


Figure 40: Fall PHY 121 responses to the question in Figure 37; N=98

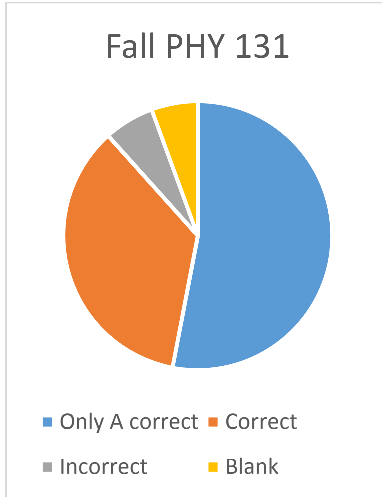


Figure 41: Fall PHY 131 responses to the question in Figure 37; N=433

The charts above are the results for solving for x, but the results for y are almost identical. Notice that many students were able to solve part A, but very few students could solve part B. Many students had struggles like the one shown below:

$$y = \frac{ax}{b}$$

$$a \cdot x + a \cdot \frac{ax}{b} = c$$

$$ax + \frac{a^2 x}{b} = c$$

Figure 42: Example of student work failing to finish solving the system of equations

Figure 42 only shows the end of this student's work, but it shows a common roadblock that many students appeared to face. They find y in terms of x (or the other way around) and make the substitution, but then they become stuck, apparently unable to realize that they need to use the distributive property to get x (or y) by itself. This suggests that there may be an algebraic "toolbox" of techniques like the distributive property that physics students use to solve problems but that many students forget to use. Other possible candidates include the commutative property and substitution. It may be beneficial for future work to keep looking for other algebraic tools that belong in every physicists "toolbox."

Additional Observations

There are a few other stray observations made while conducting research that do not fit in the results section but still deserve some mention. First, there was a summer semester diagnostic given to a single PHY 121 class. However, it is barely mentioned in this thesis because the results were far lower than any other class that took any of the diagnostics. The questions were identical or of comparable difficulty, but the percentages of correct answers were far lower. Below is a question that was on all three diagnostics and the results for each (only considering the PHY 121 students).

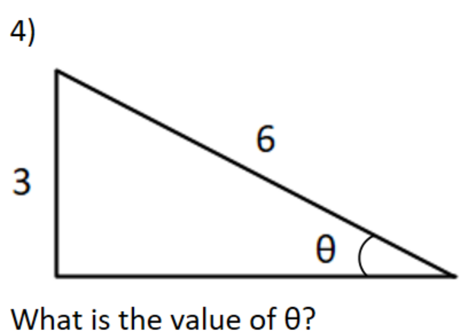


Figure 43: Question that appeared on all diagnostics, including the summer diagnostic; Correct Answer: 30°

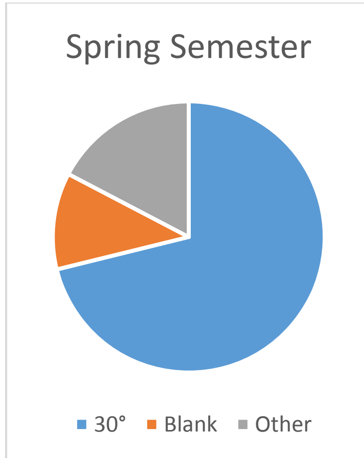


Figure 44: Spring PHY 121 responses to the question in Figure 43; N=104

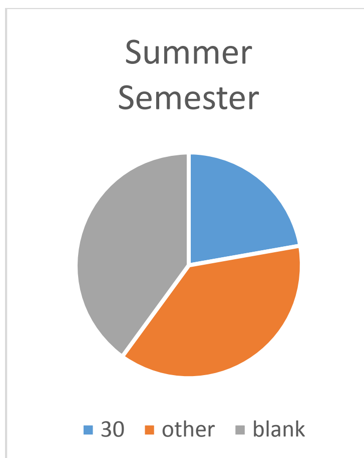


Figure 45: Summer PHY 121 responses to the question in Figure 43; N=45

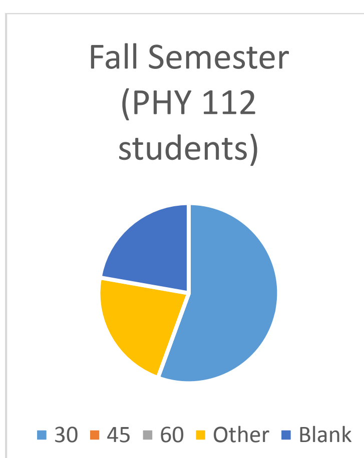
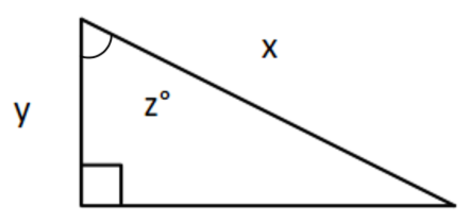


Figure 46: Fall PHY 112 responses to the question in Figure 43; N=54

(In the fall semester, PHY 121 students only answered a multiple choice version, so the 112 response has been included for comparison instead. The two classes had similar results success rates, so this should be acceptable.) The instructors for these courses also noticed a significant difference between the preparedness of the summer semester and the fall and spring semester classes. This issue could be the focus of future studies.

The final issue of note is the performance of the different classes. Without a doubt, PHY 111 had the lowest success and PHY 131 had the highest; this is to be expected. PHY 112 and PHY 121, though, were on more even footing. PHY 112 students have the advantage of already taken a semester of physics, but PHY 121 students will have taken calculus and should have a higher level of mathematical ability. The results from the diagnostics show they are about even. In the spring semester, PHY 121 scored better than 112 on all 11 problems. In the fall, however, PHY 112 actually scored higher than PHY 121 on several problems. The following is just one example where the 112 students had a more correct answers ($x = \frac{y}{cosz}$) than the 121 students.

3)



What is the value of x?

Figure 47: Question involving trigonometry on the fall diagnostic; Correct answer: $x = \frac{y}{\cos z}$

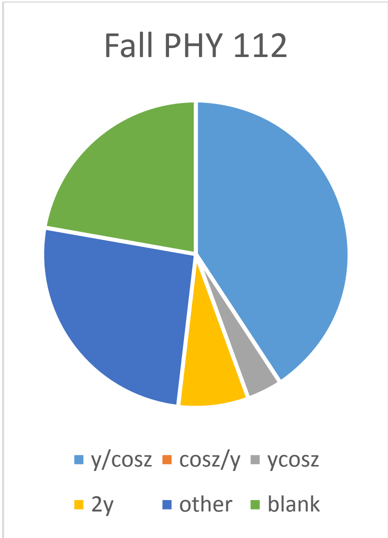


Figure 48: Fall PHY 112 responses to the question in Figure 47; N=54

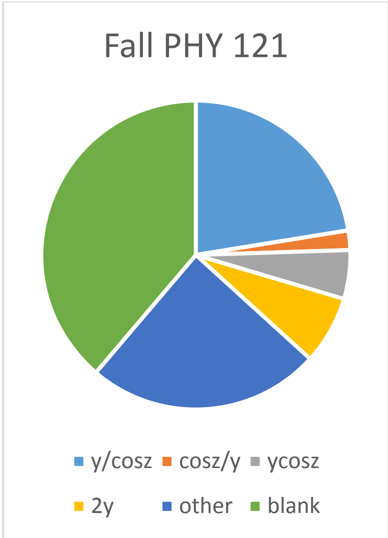


Figure 49: Fall PHY 121 responses to the question in Figure 48; N=98

It should also be noted that all of the students in PHY 112 and PHY 121 that took the diagnostic were from the Polytechnic campus. The only Tempe students to take any diagnostic were the fall PHY 131 students, so there is no worry that this difference is caused by a difference between the two campuses. These results are not especially surprising or useful for this thesis, but it is an interesting result nonetheless.

Conclusion

It is well documented that there are many areas of mathematics that physics students struggle with. This thesis, as well as many other papers, have documented student difficulties with algebra, vectors, and trigonometry. Despite being vital to all physics courses, many students are not able to solve basic right triangle trigonometry problems. A substantial number of students struggle with the most basic vector addition and direction problems. Many students do not know or do not consider some of the algebraic tools that they may need to solve problems, such as the distributive property. Symbolic notation and word problems also seem to make otherwise easy problems more complex. Finally, there is evidence that many of students' mistakes are just the result of carelessness.

There are many strategies that students use to solve these math problems, some more successful than others. When solving vector problems, considering the component method provides better results than the arrow (or tip-to-tail) method in both addition and direction problems. In addition, when solving trigonometry problems, it seems that experience and comfort with the trigonometric functions yields far better results than simply memorizing SOH CAH TOA. Lastly, simply reviewing one's work has the potential to eliminate up to 25% of errors.

In conclusion, there are many areas of mathematics that students struggle with, including vectors, trigonometry, and algebra. However, there are some empirically tested methods and ideas that students can use to improve their performance on these types of problems. With further research, some of the ideas outlined in this thesis could be used to develop effective instructional aids to help students understand and use the math necessary to solve problems in physics.

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