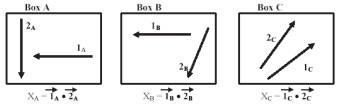
Student difficulties with graphical representation of vector products: crossing and dotting beyond *t*'s and *i*'s*

Warren M. Christensen, Ngoc-Loan Nguyen, and David E. Meltzer
Iowa State University
*Supported in part by NSF REC #0206683

In an effort to test students' understanding of the graphical representation of scalar and vector products, a four-question quiz was administered to students in a first-semester calculus-based physics course [221] during the spring and summer of 2004, as well to as students in a second semester calculus-based physics course [222] during the summer of 2004. The questions and results are below. (Questions were administered during the final week of the spring course, and near the mid-point of the summer courses.)

1. In each of the three boxes below (Box A, Box B, Box C) there is a pair of vectors, $\vec{1}$ and $\vec{2}$. All arrows have the same length. Consider the dot product ("scalar product") of each pair of vectors.

X_A is the dot product of the vectors in Box A.
X_B is the dot product of the vectors in Box B.
X_C is the dot product of the vectors in Box C.



Choose the answer that best describes the dot products: X_A, X_B, X_C.

A. $X_A = X_B = X_C$
B. $X_A > X_C > X_B$
C. $X_A > X_B > X_C$
D. $X_B > X_C > X_A$
E. $X_B > X_A > X_C$
F. $X_B = X_A > X_C$
G. $X_C > X_A > X_B$
H. $X_C > X_B > X_A$
I. $X_C > X_B = X_A$
J. Cannot be determined from the given information.

$\vec{1} \bullet \vec{2} = |\vec{1}||\vec{2}|\cos(\theta)$
Since all vectors are of equal length:
$X_i \propto \cos(\theta_i)$;
$\cos(\theta_C) > \cos(\theta_B) > \cos(\theta_A)$
$\therefore X_C > X_B > X_A$

Correct Responses

	N	% of *N*
221 Spring	168	68%
221 Summer	36	64%
222 Summer	41	76%

Students failing to recognize X_A is smallest (i.e., responding with answers A, B, C, E , F, or G):

	N	% of *N*
221 Spring	168	28%
221 Summer	36	22%
222 Summer	41	20%

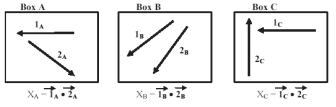
Students failing to recognize X_C is the greatest (i.e., responding with answers A, B, C, D, E, or F):

	N	% of *N*
221 Spring	168	27%
221 Summer	36	22%
222 Summer	41	17%

Those students who appeared to utilize a component method for calculating the scalar products were successful in obtaining a correct answer. Students often abandoned a component method in favor of some equation representation [i.e., $|1_A||2_A|\cos(\theta)$], with varying degrees of success.

2. In each of the three boxes below (Box A, Box B, Box C) there is a pair of vectors, $\vec{1}$ and $\vec{2}$. All arrows have the same length. Consider the dot product ("scalar product") of each pair of vectors.

X_A is the dot product of the vectors in Box A.
X_B is the dot product of the vectors in Box B.
X_C is the dot product of the vectors in Box C.



Choose the answer that best describes the dot products: X_A, X_B, X_C.

A. $X_A > 0, X_B > 0, X_C > 0$
B. $X_A > 0, X_B > 0, X_C = 0$
C. $X_A > 0, X_B > 0, X_C < 0$
D. $X_A > 0, X_B = 0, X_C > 0$
E. $X_A > 0, X_B = 0, X_C < 0$
F. $X_A < 0, X_B > 0, X_C > 0$
G. $X_A < 0, X_B > 0, X_C = 0$
H. $X_A < 0, X_B > 0, X_C < 0$
I. $X_A < 0, X_B = 0, X_C > 0$
J. Cannot be determined from the given information.

$\vec{1} \bullet \vec{2} = |\vec{1}||\vec{2}|\cos(\theta)$
Since all vectors are of equal length:
$X_i \propto \cos(\theta_i)$
$\cos(\theta_A) < 0, \cos(\theta_B) > 0, \cos(\theta_C) = 0$
$\therefore X_A < 0, X_B > 0, X_C = 0$

Correct Responses

	N	% of *N*
221 Spring	168	52%
221 Summer	36	58%
222 Summer	41	61%

Students failing to recognize X_C is zero (i.e., responding with answers A, C, D, E, F, H, or I):

	N	% of *N*
221 Spring	168	28%
221 Summer	36	17%
222 Summer	41	20%

Students failing to recognize X_A is negative (i.e., responding with answers A, B, C, D, or E):

	N	% of *N*
221 Spring	168	32%
221 Summer	36	33%
222 Summer	41	27%

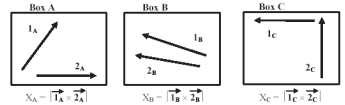
Typical student response when failing to recognize X_A is negative (seen in 221 and 222 students):

"I know C has to be 0, because cos(90) = 0, and you use the absolute values so [the magnitudes] must be >0. The angle isn't negative because it's the angle between the two vectors."

Many students chose θ to be the tip-to-tail angle, without recognizing the need to use parallel vector transport.

3. In each of the three boxes below (Box A, Box B, Box C) there is a pair of vectors, $\vec{1}$ and $\vec{2}$. All arrows have the same length. Consider the cross product ("vector product") of each pair of vectors.

X_A is the magnitude of the cross product of the vectors in Box A.
X_B is the magnitude of the cross product of the vectors in Box B.
X_C is the magnitude of the cross product of the vectors in Box C.



Choose the answer that best describes the **magnitudes** of the cross products: X_A, X_B, X_C.

A. $X_A = X_B = X_C$
B. $X_A > X_B > X_C$
C. $X_A > X_C > X_B$
D. $X_A > X_C > X_B$
E. $X_B > X_C > X_A$
F. $X_B > X_A > X_C$
G. $X_C > X_A > X_B$
H. $X_C > X_B > X_A$
I. $X_C = X_B > X_A$
J. Cannot be determined from the given information.

$|\vec{1} \times \vec{2}| = |\vec{1}||\vec{2}|\sin(\theta)$
Since all vectors are of equal length:
$X_i \propto \sin(\theta)$
$\sin(\theta_C) > \sin(\theta_B) > \sin(\theta_A)$
$\therefore X_C > X_A > X_B$

Correct Responses

	N	% of *N*
221 Spring	206	58%
221 Summer	36	50%
222 Summer	41	56%

Students failing to recognize X_C is the greatest (i.e., responding with answers A, B, C, D, E, or F):

	N	% of *N*
221 Spring	206	36%
221 Summer	36	42%
222 Summer	41	37%

Students failing to recognize X_B is smallest (i.e., responding with answers A, B, E, F, H, or I):

	N	% of *N*
221 Spring	206	35%
221 Summer	36	42%
222 Summer	41	39%

Typical student response for an incorrect calculation of the magnitude of the vector product:

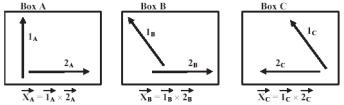
"Because for cross prod[uct] it is (1)(2)cos θ and you can factor out the (1)(2)"

Many students used a similar "cos θ" reasoning; they not only failed to recognize X_C as being the greatest quantity, but most often determined that it was zero.

Several students attempted to use a matrix method to calculate the cross product but there were no apparent successes.

4. In each of the three boxes below (Box A, Box B, Box C) there is a pair of vectors, $\vec{1}$ and $\vec{2}$. All arrows have the same length. Consider the cross product of each pair of vectors.

$\vec{X}_A$ is the cross product of the vectors in Box A.
$\vec{X}_B$ is the cross product of the vectors in Box B.
$\vec{X}_C$ is the cross product of the vectors in Box C.



Choose the answer that best describes the **direction** of the cross products: $\vec{X}_A$, $\vec{X}_B$, $\vec{X}_C$.

Note: ⊙ represents a vector pointing **out** of the page.
⊗ represents a vector pointing **into** the page.

A. $\vec{X}_A$ ⊙, $\vec{X}_B$ ⊙, $\vec{X}_C$ ⊙
B. $\vec{X}_A$ ⊙, $\vec{X}_B$ ⊙, $\vec{X}_C$ ⊗
C. $\vec{X}_A$ ⊙, $\vec{X}_B$ ⊗, $\vec{X}_C$ ⊙
D. $\vec{X}_A$ ⊙, $\vec{X}_B$ ⊗, $\vec{X}_C$ ⊗
E. $\vec{X}_A$ ⊗, $\vec{X}_B$ ⊙, $\vec{X}_C$ ⊙
F. $\vec{X}_A$ ⊗, $\vec{X}_B$ ⊙, $\vec{X}_C$ ⊗
G. $\vec{X}_A$ ⊗, $\vec{X}_B$ ⊗, $\vec{X}_C$ ⊙
H. $\vec{X}_A$ ⊗, $\vec{X}_B$ ⊗, $\vec{X}_C$ ⊗
I. Cannot be determined from the given information.

Correct Responses

	N	% of *N*
221 Spring	206	58%
221 Summer	34	53%
222 Summer	41	61%

Students responding with answer F (the directions of the vector products are reversed):

	N	% of *N*
221 Spring	206	0%
221 Summer	34	22%
222 Summer	41	20%

Students responding with answer E (all vector products are pointing out of the page):

	N	% of *N*
221 Spring	206	16%
221 Summer	34	11%
222 Summer	41	5%

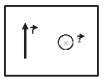
The absence of "F" responses in the spring 221 class is rather troublesome. Before the quiz was administered we speculated that F would be the most common incorrect answer. Our expectations were confirmed during the summer classes for both 221 and 222, but the absence of such responses in the spring 221 class is unexplained.

None of the students who selected response "E" provided an explanation.

One of the questions administered to the students in the Spring 221 class was given to the Summer 221 and 222 students as a question on an exam. Due to the constraints of the exam we were forced to condense the responses from 10 down to 5. The question for the 222 class was put into the context of a charged particle in a magnetic field.

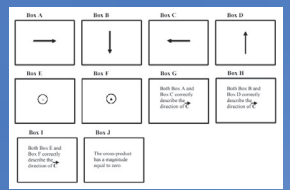
In the figure below there are two vectors, $\vec{1}$ and $\vec{2}$. There exists a cross-product ("vector product"), $\vec{C}$, of the vectors (i.e., $\vec{C} = \vec{1} \times \vec{2}$). Calculate the direction of $\vec{C}$.

Note: ⊙ represents a vector pointing **out** of the page.
⊗ represents a vector pointing **into** the page.



Which of the following boxes best describes the direction of $\vec{C}$?

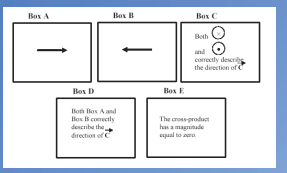
Multiple choice options for Spring 221



Spring 221 (*N* = 168)

A	B	C	D	E	F	G	H	I	J
18%	0%	40%	1%	6%	4%	17%	1%	3%	5%

Multiple choice options for Summer 221/222



Summer 221 (*N* = 48)

A	B	C	D	E
23%	50%	4%	6%	17%

Summer 222 (*N* = 56)

A	B	C	D	E
25%	68%	4%	4%	0%

In order to get down to five choices, we removed B, D, E, F, and H. Even though choices E and F had more responses than choice I, studies have shown that some students have difficulty distinguishing the direction of a vector from that of a vector in the opposite direction (Nguyen and Meltzer, 2003). The substantial number of students selecting response G seems to support that notion. Therefore, we retained response I as a choice for the summer exam question, renaming it response C.

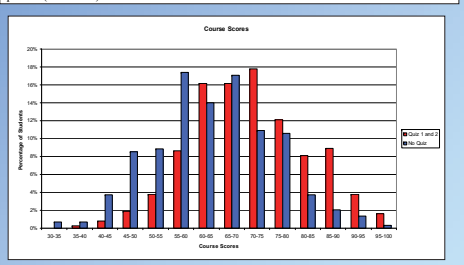
One sixth (17%) of 221 students responded that the vector product has a magnitude of zero.

On Question 3, 15% of 222 students had explicitly given "zero" for the magnitude of the vector product of two perpendicular vectors (i.e., stated that $X_C = 0$ on that question). On this exam question, by contrast, none gave that response. It is possible that the magnetic-field context of the 222 exam question was responsible for this difference.

Both 221 and 222 students seem to have significant difficulty in applying the right-hand rule, as ~25% of both classes chose the direction *opposite* to the correct response on the exam question. This is consistent with the responses to Question 4.

The biased nature of a "random" sample when using an online medium

In the process of testing students' understanding of vector and scalar products, we were offered an opportunity to use an online medium, WebCT, to administer a quiz. Complying with the instructor's request, we divided our six question quiz into two 3-question quizzes. At the end of the semester, we analyzed the overall class scores (final numerical grade) of every student in the class. Below is the score distribution for the two groups that took quizzes (combined) and the one that did not.



Statistical analysis shows the following:

Descriptives

SCORE

	N	Mean	Std. Error	95% Confidence Interval for Mean	
				Lower Bound	Upper Bound
No Quiz	293	63.8	.687	62.4	65.1
Quiz 1	167	71.3	.844	69.6	72.9
Quiz 2	204	70.9	.818	69.3	72.6

The mean course score for students who took Quiz 1 (71.3) is statistically identical to the score of those who took Quiz 2 (70.9), but significantly larger ($p < 0.0001$) than that of those who took no quiz (63.8) [a difference equivalent to one full letter grade].