Identifying and Addressing Students' Mathematical Difficulties in Introductory Physics Courses

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Supported in part by NSF DUE #1504986

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in collaboration with

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and

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The Problem

- Difficulties with very basic math skills impact performance of introductory physics students.
- The difficulties are often not resolved by students' previous mathematical training.
- Students can't effectively grapple with physics ideas when they feel overburdened in dealing with calculational issues.
- (Therefore:) Physics instructors need to deal with their students' mathematical difficulties

Outline

- Prevalence and nature of difficulties with trigonometry
- Examples of difficulties with vectors, and effective remedies
- Difficulties with algebraic symbols and operations
  – physics problems posed in symbolic form
  – physics students' algebraic difficulties
- Implications for instruction

Work to Date

- Administer (and analyze) written diagnostic, given to 1400 students in 18 algebra- and calculus-based physics classes over four semesters at Arizona State University during 2016-2017

- Carry out individual interviews with 65 students enrolled in those or similar courses during same period

Difficulties with Trigonometry

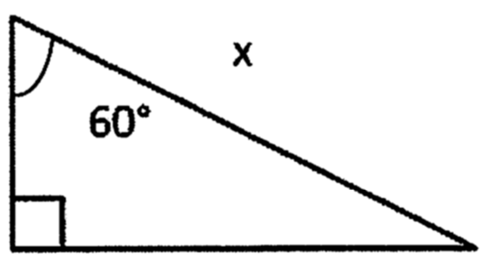
- Many students are confused or unaware (or have forgotten) about the relationships between sides and angles in a right triangle.

- *Examples:* Questions from a diagnostic math test administered at Arizona State University, 2016-2017 (Administered as no-credit quiz during first week labs and/or recitation sections; **calculators allowed**)

Trigonometry Questions

with samples of correct student responses

1.



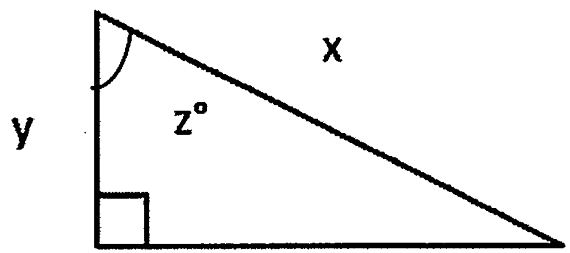
What is the value of x?

$$\cos 60 = \frac{10}{x}$$

$$x \cos 60 = 10$$

$$x = \frac{10}{\cos 60} = 20$$

2.



cos z = s/x

What is the value of x?

A. ycos(z)

B. ycos(z)sin(z)

C. y/sin(z)

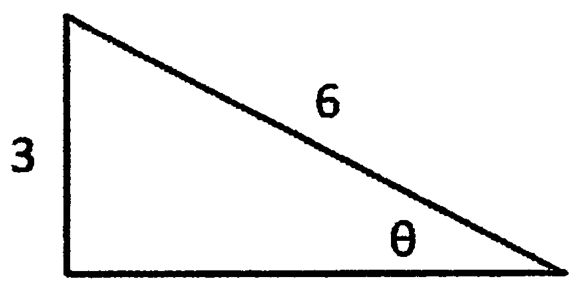
D. ysin(z)

E. ycos(z)/sin(z)

F. y/cos(z)

G. None of the above________________________

3.



What is the value of θ?

$$\sin^{-1}(\theta) = \sin^{-1}\left(3/6\right)$$

$$\theta = 30^{\circ}$$

Trigonometry Questions:

Correct Response Rate, #1-3 combined

ASU Polytechnic campus, Spring + Fall average:

Algebra-based course, 1st semester, (N = 116): 37%

Algebra-based course, 2nd semester, (N =79): 48%

ASU Polytechnic campus, Spring (2-year average):

Calculus-based course, 1st semester, (N = 146): 66%

➡ $\frac{1}{3}$ *to* $\frac{2}{3}$ *of students confused on basic trigonometry relations*

Results on Trigonometry Questions

Errors observed:

(i) use of incorrect trigonometric function (e.g., cosine instead of sine), or misunderstanding of definition;

(ii) unaware (or forgot) about inverse trigonometric functions, e.g., arctan, arcsin, arccos [$\tan^{-1}$, $\sin^{-1}$, $\cos^{-1}$]

- **How to address these problems:** It seems that many students require substantial additional *practice and repetition* with basic trigonometric procedures.

Factors Affecting Results

Differences in results observed between:

- Spring- and fall-semester offerings of same course
- Offerings of same course on different campuses (Polytechnic campus in Mesa, main campus in Tempe)
- Different type of courses (algebra- and calculus-based) and different semesters of same course (first-semester and second-semester)
- Multiple-choice and non-multiple-choice versions of same questions

Trigonometry Questions:
Spring/Fall Semester Difference

Error Rate (% incorrect responses) [*$p < 0.05$]

Algebra-based course, first semester; #1-3 combined:

ASU Polytechnic campus, Spring ($N = 72$): 67%

ASU Polytechnic campus, Fall ($N = 44$): 58%

Algebra-based course, second semester; #1-3 combined:

ASU Polytechnic campus, Spring ($N = 52$): 59%

ASU Polytechnic campus, Fall ($N = 27$): 44%

**Calculus-based course, first semester; #1 only:*

ASU Polytechnic campus, Spring ($N = 104$): 40%

ASU Polytechnic campus, Fall ($N = 98$): 56%

Trigonometry Questions:
Spring/Fall Semester Difference

Error Rate (% incorrect responses) **[*$p < 0.05$]**

Algebra-based course, first semester; #1-3 combined:

ASU Polytechnic campus, Spring ($N = 72$): 67% ⇐

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Spring/Fall Semester Difference

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ASU Polytechnic campus, Spring ($N = 104$): 40%

ASU Polytechnic campus, Fall ($N = 98$): 56% ⟸

Trigonometry Questions:
Polytechnic/Tempe Campus Difference

Error Rate (% incorrect responses) [*p = 0.01]

**Algebra-based course, second semester; #1-3 combined:*

ASU Polytechnic campus, Spring (N = 52): 59%

ASU Tempe campus, Spring (N = 61): 35%

Trigonometry Questions:
Course Difference

Error Rate (% incorrect responses), #1 only **[*$p < 0.05$]**

ASU Polytechnic campus, Spring + Fall average:

Algebra-based course, 1st semester, (N = 166): 59%

* Algebra-based course, 2nd semester, (N = 106): 53%

Calculus-based course, 1st semester, (N = 202): 48%

Trigonometry Questions:

Multiple-Choice vs. Non-Multiple-Choice

(Higher Error Rate on Non-Multiple-Choice [Non-MC])

Error Rate Difference (% incorrect responses), Non-MC−MC

Course #1, Problem #2: +15
Course #1, Problem #3: +18

Course #2, Problem #2: +9
Course #2, Problem #3: +9

Course #3, Problem #2: +5
Course #3, Problem #3: +34

Course #4, Problem #2: +10
Course #4, Problem #3: +5

Trigonometry Questions: Summary

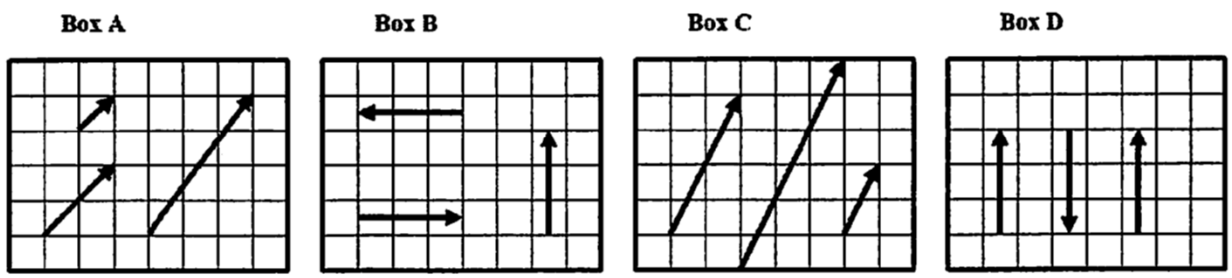
- Regardless of course, semester, campus, or question type, between 30% and 70% of introductory physics students at ASU have significant difficulties with basic trigonometric relationships.
- Students frequently tended to self-correct errors during interviews, suggesting that many of the errors were “careless” or due to insufficient review or practice.

Difficulties with Vector Concepts

- **Vector Concepts:** Many students are confused about the fundamental meaning of vector *direction*, as well as the role of direction in determining the resultant of two or more vectors (Nguyen and Meltzer, 2003; Barniol and Zavala, 2014).

Vector Direction Question

7. In the four boxes below are collections of vectors on top of equally spaced grid lines. Choose the answer from the list below that most correctly describes the comparative **directions** of the vectors within each box.



Possible answers. Select the best one.

A. Box **A** has all vectors with the same direction
B. Box **B** has all vectors with the same direction
C. Box **C** has all vectors with the same direction
D. Box **D** has all vectors with the same direction
E. Both boxes **A** and **C** have vectors that all have the same direction
F. Both boxes **A** and **D** have vectors that all have the same direction
G. Both boxes **C** and **D** have vectors that all have the same direction
H. The boxes, **A**, **C**, and **D** have vectors that all have the same direction
I. None of the above boxes have vectors with the same direction

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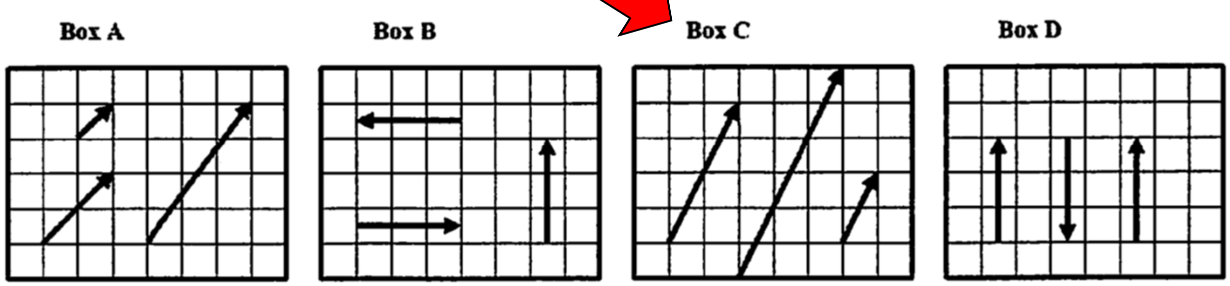

Possible answers. Select the best one.

All same direction

A. Box **A** has all vectors with the same direction
B. Box **B** has all vectors with the same direction
(C.) Box **C** has all vectors with the same direction
D. Box **D** has all vectors with the same direction
E. Both boxes **A** and **C** have vectors that all have the same direction
F. Both boxes **A** and **D** have vectors that all have the same direction
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H. The boxes, **A**, **C**, and **D** have vectors that all have the same direction
I. None of the above boxes have vectors with the same direction

Results on Vector Direction

Percent Correct Responses

Calculus-based physics, second semester: **66%** (*N* =29)

Calculus-based physics, first semester: **51%** (*N* = 104)

Algebra-based physics, second semester: **40%** (*N* = 52)

Algebra-based physics, first semester: **40%** (*N* = 72)

Vector Direction, Most Common Error

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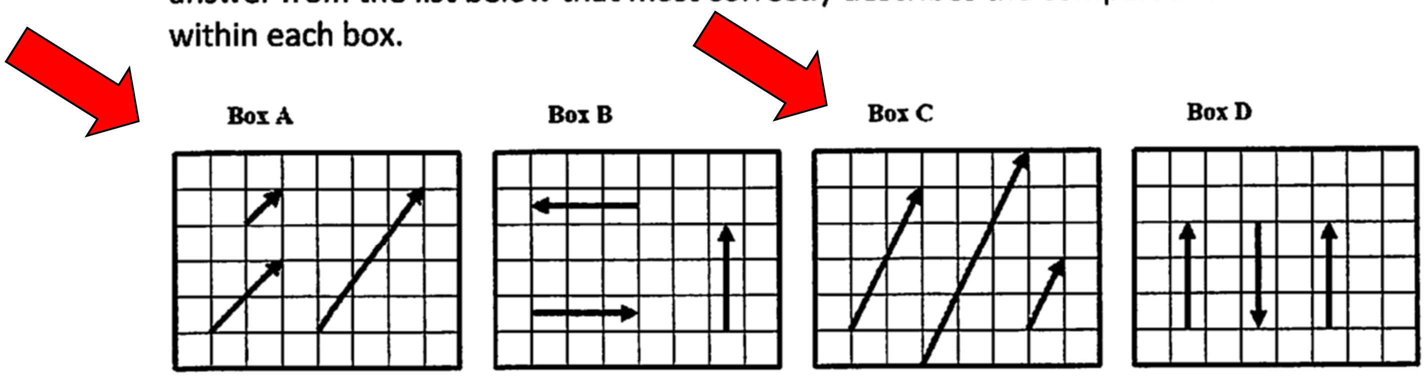

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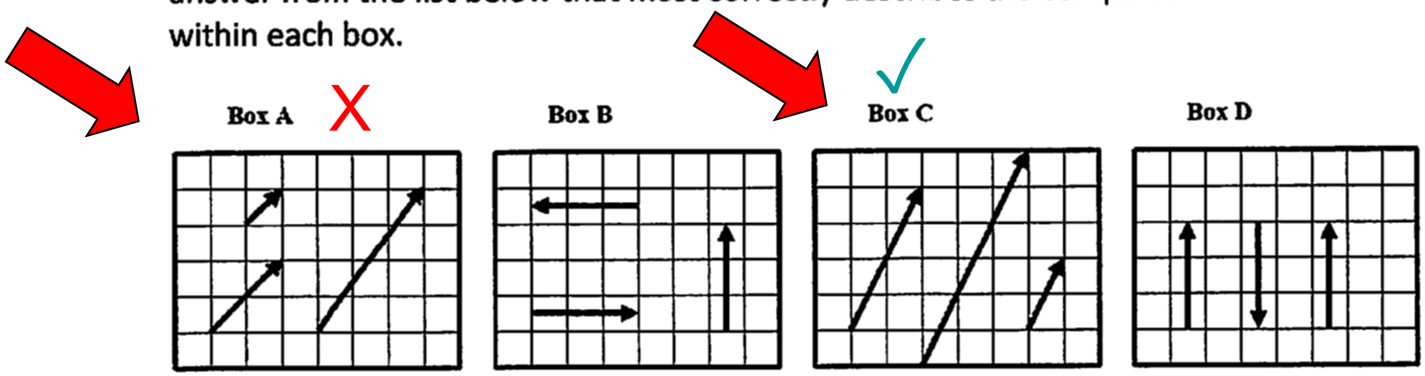

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I. None of the above boxes have vectors with the same direction

Vector Addition (Graphical)

- Students have difficulty interpreting and manipulating vector quantities represented as arrows [Nguyen and Meltzer, 2003; Barniol and Zavala, 2014)

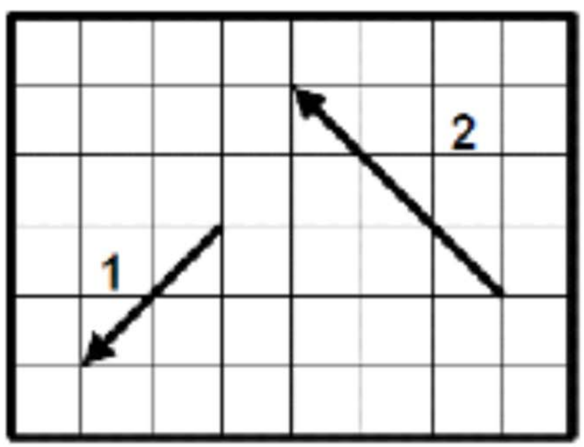
Example: Add (or subtract) vectors A and B to find the resultant

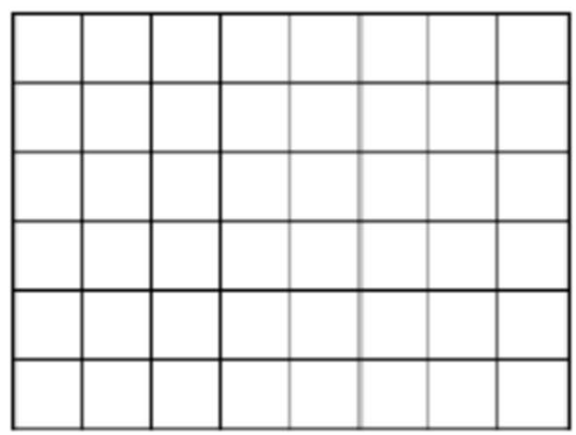
Addition of Vectors

6)

In the figure below there are two vectors $\overline{\mathbf{1}}$ and $\overline{\mathbf{2}}$. In the empty grid, draw the sum or vector addition $\overline{\boldsymbol{R}}$ of the two (i.e., $\overline{\boldsymbol{R}} = \overline{\mathbf{1}} + \overline{\mathbf{2}}$).

Note: You can draw other vectors in the empty grid, but be sure to label $\overline{\boldsymbol{R}}$ clearly.





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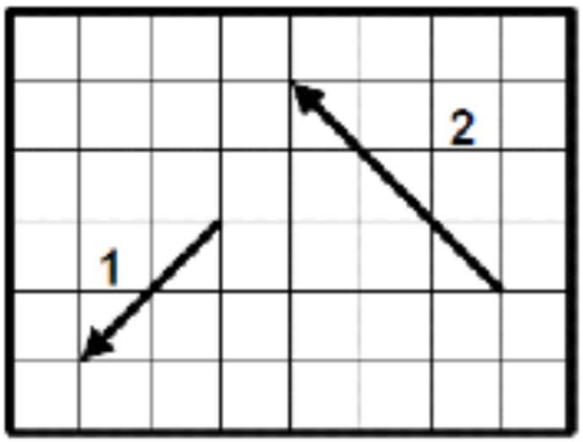
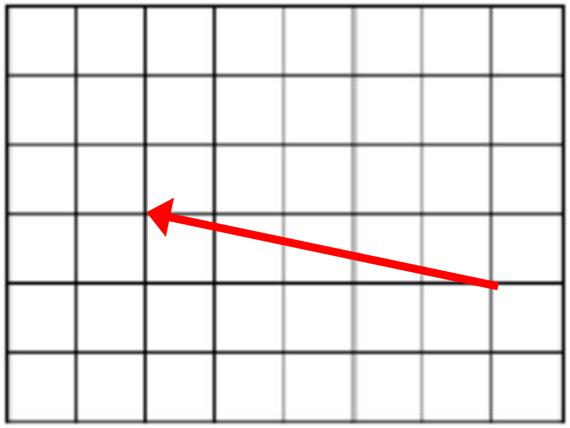




Addition of Vectors

Percent Correct Responses

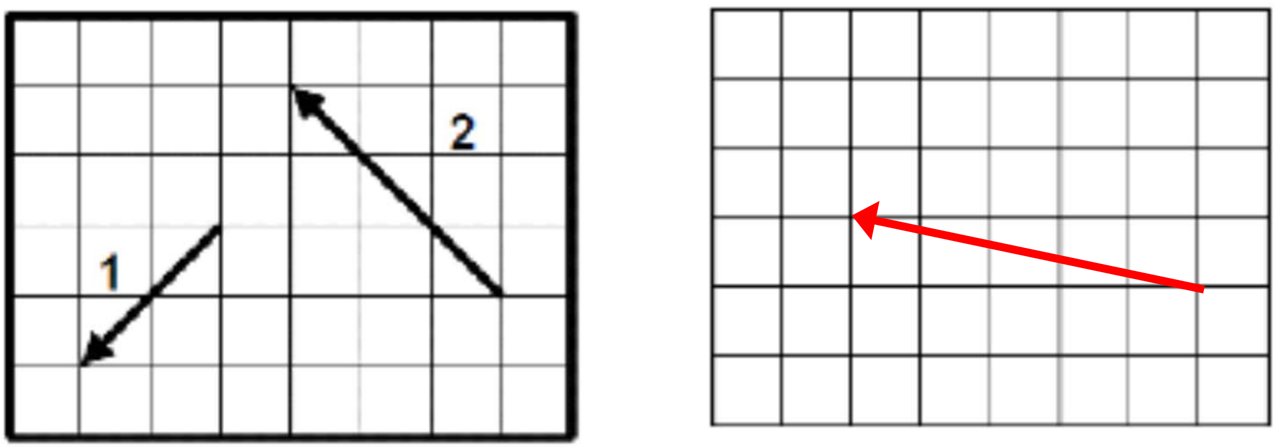
Algebra-based physics, second semester: **36%** (*N* = 61)

Calculus-based physics, first semester: **71%** (*N* = 140)

6)

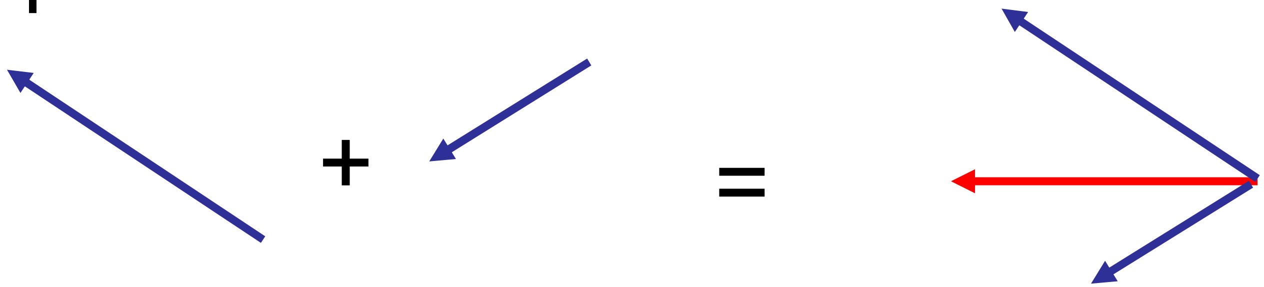
In the figure below there are two vectors $\overline{\mathbf{1}}$ and $\overline{\mathbf{2}}$. In the empty grid, draw the sum or vector addition $\overline{\boldsymbol{R}}$ of the two (i.e., $\overline{\boldsymbol{R}} = \overline{\mathbf{1}} + \overline{\mathbf{2}}$).

Note: You can draw other vectors in the empty grid, but be sure to label $\overline{\boldsymbol{R}}$ clearly.

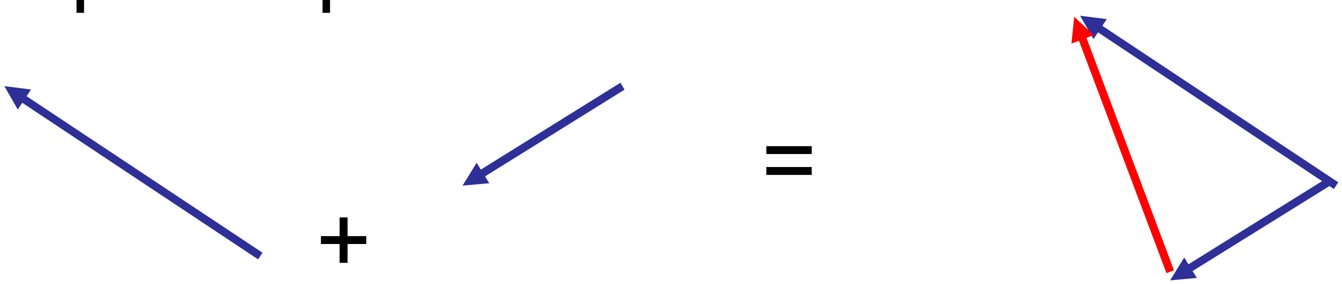


Common Student Errors With Vector Addition

- “Split the Difference” or “Bisector Vector”:



- “Tip-to-Tip”:



Difficulties with Vector Addition

How to address these difficulties:

- Practice with a variety of vector orientations; introduce and use the "*ijk*" coordinate representation for vectors (Heckler and Scaife, 2015)
- Design tutorial worksheet to aid students' understanding of scalar ("dot") product of vectors (Barniol and Zavala, 2016)
- Provide extensive on-line practice and homework assignments related to frequently used vector procedures (Mikula and Heckler, 2017)

Physics Students' Difficulties with Algebraic Symbols and Operations

- Extensive investigations by Torigoe and Gladding (2007; 2007; 2011): Probed differences in University of Illinois students' responses to physics problems posed in numerical and symbolic form.
  - In general, students tended to have more difficulties with questions in symbolic form.
- Our investigation at Arizona State probed physics students' responses to mathematical problems stripped of all physics context

Torigoe and Gladding (2011), Findings

1. Significantly higher proportion of correct responses on *some* types of numerical questions, not on others
2. Difference was greater for students in bottom quarter of class
3. Larger difference linked to difficulties with multiple and simultaneous equations, symbol confusion, and misuse of compound expressions.

Students' Difficulties with Symbols

Confusion of symbolic meaning: Students perform worse on solving problems when symbols are used to represent common physical quantities in equations, e.g., "*m*" instead of "1.5 kg" [Torigoe and Gladding, 2007; 2011)

Example [University of Illinois]:

Version #1: A car can go from 0 to 60 m/s in 8 s. At what distance *d* from the start at rest is the car traveling 30 m/s? *[93% correct]*

Version #2: A car can go from 0 to v_1 in t_1 seconds. At what distance *d* from the start at rest is the car traveling $(v_1/2)$? *[57% correct]*

Much worse!

Difficulties with Symbols

(Torigoe and Gladding, 2007; 2011)

- Confusion between similar symbols, e.g., using v for v_1 and vice-versa.
- Distinguishing between known and unknown information when *both* are represented by symbols
- Misunderstanding compound expressions, e.g., failing to specify $m = 3M$ or $v = v_0/2$ in the relevant general equation involving m or v.

From Torigoe & Gladding (2011):

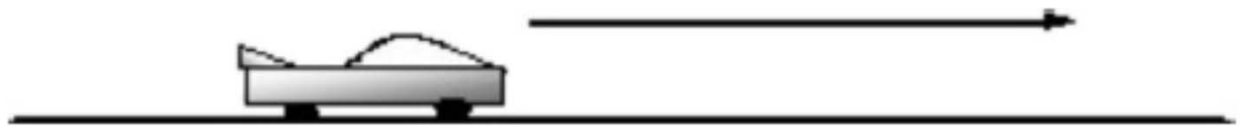


Fig. 5. Diagram for question 2.

Question 2 (numeric). A car can go from 0 to 60 m/s in 8 s. At what distance *d* from the start (at rest) is the car traveling 30 m/s? [Assume a constant acceleration (see Fig. 5).]

Question 2 (symbolic). A car can go from 0 to v_1 in t_1 seconds. At what distance *d* from the start (at rest) is the car traveling $(v_1/2)$? [Assume a constant acceleration (see Fig. 5).]

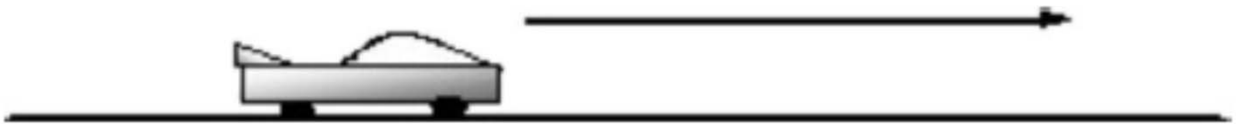
(numeric)

(a) 30 m
(b*) 60 m
(c) 120 m
(d) 240 m
(e) 480 m

(symbolic).

(a) $d=v_1t_1$
(b) $d=v_1t_1/2$
(c) $d=v_1t_1/4$
(d*) $d=v_1t_1/8$
(e) $d=v_1t_1/16$

From Torigoe & Gladding (2011):



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Question 2 (symbolic). A car can go from 0 to v_1 in t_1 seconds. At what distance d from the start (at rest) is the car traveling $(v_1/2)$? [Assume a constant acceleration (see Fig. 5).]

Sample solution for symbolic version:

$v^2 = v_0^2 + 2ad$

$v_0 = 0$

$\rightarrow v^2 = 2ad$

$\rightarrow d = v^2/2a$

$a = \Delta v/\Delta t = v_1/t$

$v = v_1/2$

$\Rightarrow$

$d = v^2/2a$

$= (v_1/2)^2/2a$

$= (v_1^2/4)/2(v_1/t)$

$= (v_1^2/4)/(2v_1/t)$

$= v_1 t_1/8$

From Torigoe & Gladding (2011):

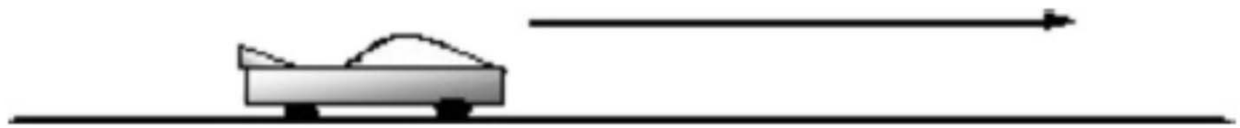


Fig. 5. Diagram for question 2.

Question 2 (numeric). A car can go from 0 to 60 m/s in 8 s. At what distance d from the start (at rest) is the car traveling 30 m/s? [Assume a constant acceleration (see Fig. 5).]

"Student" solution for numeric version:

(1) $a = \Delta v/\Delta t = 60/8 = 7.5$

(2) $v^2 = v_0^2 + 2ad$

$v_0 = 0$

$\rightarrow v^2 = 2ad$

$\rightarrow (30)^2 = 2(7.5)d$

$900 = 15d$

$900/15 = d$

60 m = d

No algebra needed!

Results on #2

[Torigoe and Gladding, 2007; 2011]

- **Numeric version:** 93% correct ($N \approx 380$)
- **Symbolic version*:** 57% correct ($N \approx 380$)

*numerical values of v_1 and t_1 *not* provided

➡ *Highly significant difference*

Common errors on symbolic version:

- Specification of $a = \frac{(\frac{v_1}{2})}{t}$ instead of $a = \frac{(v_1)}{t}$
- Final answer *$(v_1 t_1/2)$ and $(v_1 t_1/4)$* instead of *$(v_1 t_1/8)$*

Does complexity of expression play a role?

In symbolic version, have $d = (v_1^2/4)/(2v_1/t)$

→ *simplify to* $d = v_1 t_1/8$

Compared to, in numeric version:

$d = 900/15$

→ $d = 60$

Not equivalent operations!

Does complexity of expression play a role?

- Probably, but evidence so far is not clear
- Torigoe and Gladding noted confusion regarding *meaning of symbols*, not necessarily with algebraic manipulations

Algebra: Simultaneous Equations

Algebra: Simultaneous Equations

- Do differences in students' success rate between numerical and symbolic versions of same problem persist when simultaneous equations are involved? (E.g., two equations, two unknowns)

From Torigoe and Gladding (2011):

$$Mg - T = Ma$$
$$TR = I\alpha$$
$$[I = \tfrac{1}{2} MR^2;\ \alpha = a/R]$$

$\ldots \rightarrow$

$$Mg - T = Ma$$
$$TR = [\tfrac{1}{2} MR^2][a/R]$$
$$a = ?$$

$$F_{net} = ma$$
$$\tau_{net} = I\,\alpha$$

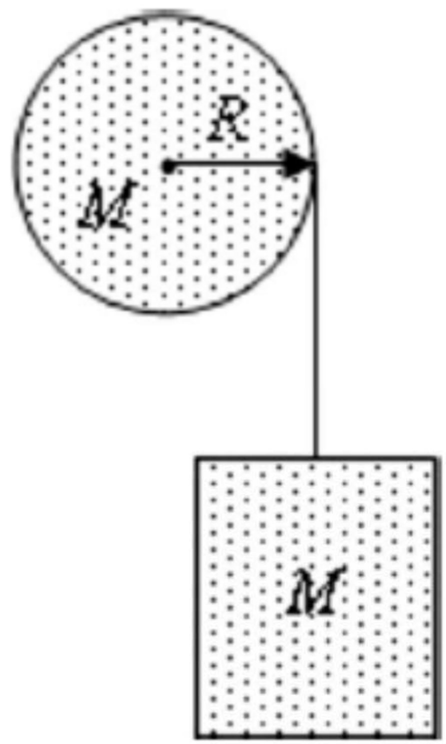


Fig. 7. Diagram for question 10.

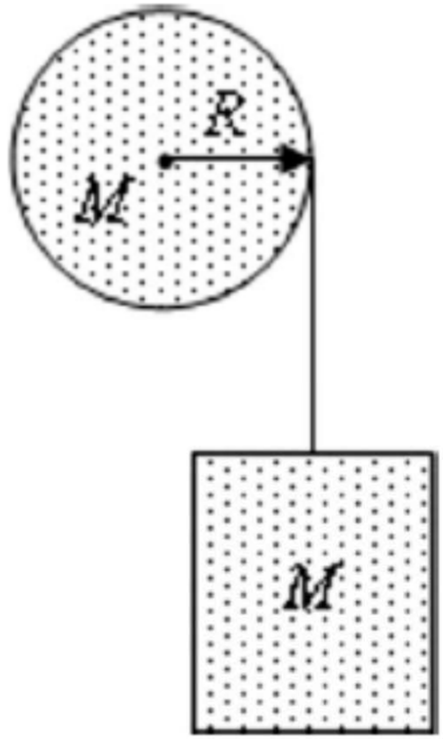
Question 10 (numeric). A uniform disk of mass $M = 8$ kg and radius $R = 0.5$ m has a string wound around its rim. The disk is free to spin about a pin through the center of the disk. A mass $M = 8$ kg (same mass as the disk) is connected to the string and is dropped from rest. What is the acceleration a of the block? (See Fig. 7.)

From Torigoe and Gladding (2011):

$Mg - T = Ma$
$TR = I\alpha$
$[I = ½ MR^2; \alpha = a/R]$

…→

$Mg - T = Ma$
$TR = [½ MR^2][a/R]$
$a = ?$



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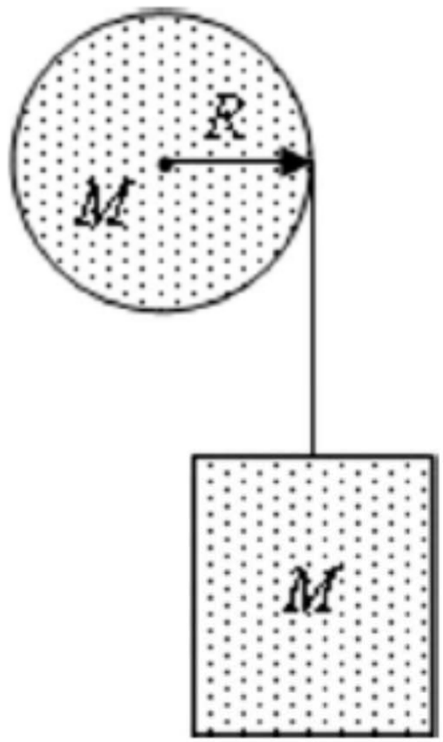


Fig. 7. Diagram for question 10.

Question 10 (numeric). A uniform disk of mass M [redacted] and radius R [redacted] has a string wound around its rim. The disk is free to spin about a pin through the center of the disk. A mass M [redacted] (same mass as the disk) is connected to the string and is dropped from rest. What is the acceleration a of the block? (See Fig. 7.)

$$Mg - T = Ma$$
$$TR = I\alpha$$
$$[I = ½\, MR^2;\ \alpha = a/R]$$

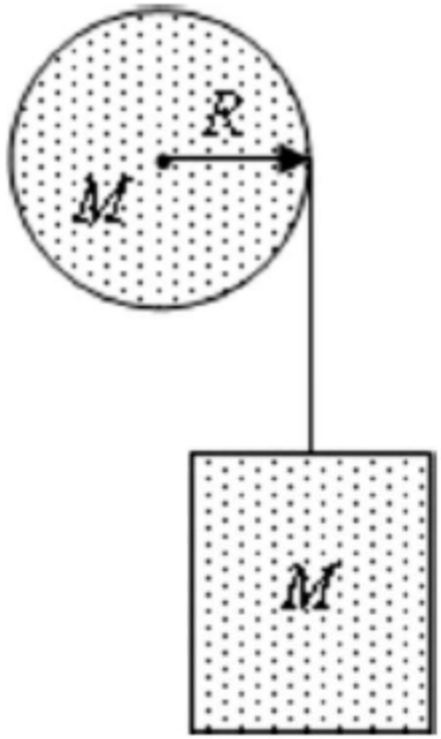
…→

$$Mg - T = Ma$$
$$TR = [½\, MR^2][a/R]$$
$$a = ?$$

$$\mathrm{a} - y = \mathrm{b}x$$
$$\mathrm{c}y = \mathrm{d}x$$
$$x = ?$$

Symbolic version

From Torigoe and Gladding (2011):



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$Mg - T = Ma$
$TR = I\alpha$
$[I = ½ MR^2;\ \alpha = a/R]$

…→

$Mg - T = Ma$
$TR = [½ MR^2][a/R]$
$a = ?$

$78.4 - y = 8x$
$0.5y = 2x$
$x = ?$

Numeric version

Question 10 (numeric). A uniform disk of mass $M=8$ kg and radius $R=0.5$ m has a string wound around its rim. The disk is free to spin about a pin through the center of the disk. A mass $M=8$ kg (same mass as the disk) is connected to the string and is dropped from rest. What is the acceleration a of the block? (See Fig. 7.)

Results on #10

(Two equations, two unknowns)

[Torigoe and Gladding, 2011]

- **Numeric version:** 49% correct ($N \approx 380$)
- **Symbolic version*:** 53% correct ($N \approx 380$)

  *values of *M* and *R* *not* provided

⇨ *No significant difference*

("…because students are forced to use the same procedure to solve both the numeric and symbolic versions." Torigoe and Gladding, 2011)

Our Findings: Significant Differences between Numerical and Symbolic Versions of Simultaneous-Equations Problem

Summary of the Data

- *Algebra-based course:* 70% correct on numerical version, 20% on symbolic version;
- *Calculus-based course:* 70% correct on numerical version, 43% correct on symbolic version.

Algebra: Simultaneous Equations

$3x = 2y$
$5x + y = 26$

What are the values of x and y? Show all your steps. For example, $x = 2$, $y = 5$ (These are NOT the correct answers).

Correct Response Rate, ASU (% correct responses)

Algebra-based course, second semester (N = 123): 70%

[This is the “numerical” problem]

Algebra: Simultaneous Equations

$x \cos(20°) = y \cos(70°)$
$x \cos(70°) + y \cos(20°) = 10$

What are the values of *x* and *y*? Show all your steps. Note: The value for *x* should NOT include *y*, and the value for *y* should NOT include *x*.

Correct Response Rate, ASU (% correct responses)

Algebra-based course, second semester (N =150): 20-30%
(different campuses, slightly different versions)

Algebra: Simultaneous Equations

$ax = by$
$bx + ay = c$

a, b, and c are constants.
What are the values of *x* and *y* in terms of a, b, and c? Show all your steps. Note: The value for *x* should NOT include *y*, and the value for *y* should NOT include *x*.

Correct Response Rate, ASU (% correct responses)

Algebra-based course, second semester (N =150): 10-20%
(different campuses, slightly different versions)

Only 10-20% correct responses!

Algebra: Simultaneous Equations

$a \cdot x = b \cdot y$
$b \cdot x + a \cdot y = c$

a, b, and c are constants.
What are the values of x and y in terms of a, b, and c? Show all your steps. Note: The value for x should NOT include y, and the value for y should NOT include x.

$$x = \frac{by}{a}$$

$$b\left(\frac{by}{a}\right) + ay = c$$

$$\frac{b^2 y}{a} + ay = c$$

$$y\left(\frac{b^2}{a} + a\right) = c$$

$$y = \frac{c}{\left(\frac{b^2}{a} + a\right)}$$

$$x = \frac{b\left(\frac{c}{(b^2/a + a)}\right)}{a}$$

Sample of Correct Student Response

Difficulties with Equations

- Interviews indicate that students who missed the first (numerical) problem had fundamental difficulties with arithmetic and/or algebra (e.g., failing to isolate variables, failing to substitute expression from first equation into the second equation).
- Many students who could solve the first (numerical) problem failed on one or both of the other two.

Simultaneous Equations Questions in Calculus-based course, first semester (Polytechnic campus) [N = 91]

$3x = 2y$

$5x + y = 26$

What is the numerical value of x?

70% correct

In the two equations below, a, b, c, and d represent (unknown) numbers, e.g., 3, 8, 9, 14.

$ax = by$

$cx + y = d$

$x = ?$

(Your answer for x should have a, b, c, and d in it, but *not* y.)

43% correct

Numerical vs. Algebraic

- Providing numerical versions of problems allows students to side-step difficulties with symbolic representation and algebraic operation
  - Problems can often be solved with purely arithmetic, non-algebraic operations
- However, certain types of problems ("equation-priority": Torigoe and Gladding, 2011) *require* algebraic operations for solution, for example:
  - simultaneous equations;
  - problems posed in purely symbolic form;
  - problems of form $Ax + B = Cx + D$

Other Difficulties with Symbols

- Possible confusion due merely to replacing numbers by symbols
- Is this a real difficulty for physics students?

Confusion due to replacing numbers by symbols

$$3 = \frac{5}{x}$$

$$x = ?$$

$$a = \frac{b}{x}$$

$$x = ?$$

$$\cos 60° = \frac{10}{x}$$

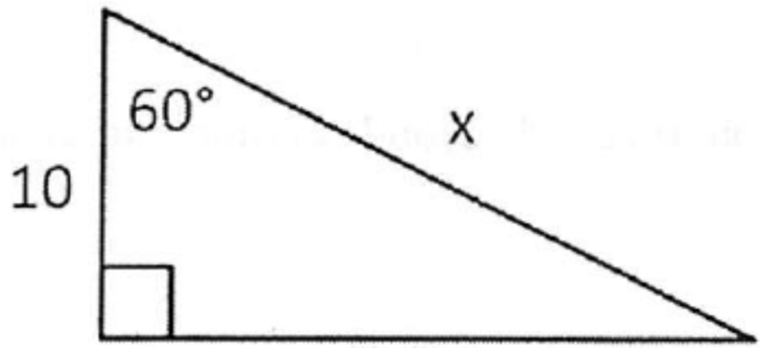
$$x = ?$$

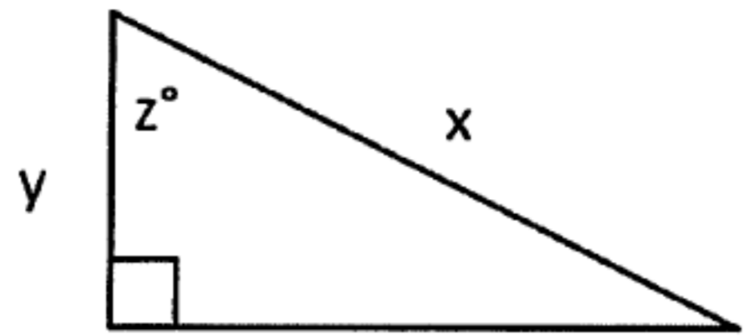
$$\cos z° = \frac{y}{x}$$

$$x = ?$$

"Level 0": Confusion due to replacing numbers by symbols

What is the value of *x*?





49% correct

41% correct

[First-semester (Fall 2017), calculus-based; $N = 91$]

McNemar Test for Correlated Proportions: $p = 0.10$

Difficulties with Mathematical Operations

(Booth et al., 2014)

- Negative Sign, e.g., moving a term without changing its sign; deleting or adding a negative sign;
- Equality, e.g., performing operations without maintaining balance on both sides of an equals sign;
- Mathematical Property, e.g., inappropriately applying the distributive property;
- Fraction, e.g., moving a term from the numerator to the denominator or vice versa

➡ All of these observed in our investigation

Why the Difficulties with Symbols?

Some Suggestions Arising from the Interviews

- In elementary math courses, “simplified forms” of equations are emphasized (i.e., few messy symbols and functions)
- Students get “overloaded” by seeing all the variables, and are unable to carry out procedures (e.g., multiplying each term in an expression by a constant [symbol]) that they do successfully with numbers (e.g., multiply through by a number)
- Other procedural failures that occur more often with symbols: cancellation, factoring out a constant, retaining coefficients from one line to the next

Sources of Difficulties

- Carelessness
  - Students *very frequently* self-correct errors during interviews
  - Evidence of carelessness on written diagnostic
- Skill practice deficit: Insufficient repetitive practice with learned skills
  - E.g., applying definitions of sine and cosine; factoring out variables in algebraic expressions
- Inability to efficiently access previous learning

Implications for Instruction

- **Vectors**
  - Essential for study of physics, and arguably *not* covered in a suitable manner during high school, or in college math courses
  - Consist of a relatively small and well-defined area of knowledge insofar as related to introductory physics
  - Difficulties addressed effectively by short-term, carefully targeted in- and out-of-class tutorials and assignments in physics courses (e.g., Barniol and Zavala (2016), Mikula and Heckler (2017)

Implications for Instruction

- **Trigonometry**
  - Essential, to some extent, for study of physics, and presumably covered thoroughly during students' previous preparation
  - Consists of a relatively small and well-defined area of knowledge insofar as related to introductory physics
  - Difficulties might be addressed effectively by short-term, carefully targeted in- and out-of-class tutorials and assignments in physics courses, designed to refresh students' previously learned knowledge and skills

Implications for Instruction

- **Algebra**
  - Essential, to some extent, for study of physics, and covered very extensively during students' previous preparation
  - Consists of a relatively large and poorly defined area of knowledge insofar as related to introductory physics
  - Difficulties might best be addressed by guiding students to (1) explicitly identify known and unknown variables; (2) carefully check and re-check key steps in calculation, to minimize careless errors; (3) slow down, review, and re-solve when possible (through alternative solution path) lengthy and/or complex calculations

Summary

- Some mathematical difficulties relate to relatively narrow, well-defined procedures and may be subject to significant improvement by short-term, targeted instruction (e.g., vectors, trigonometry)
- Other difficulties are likely more long-standing, resistant to quick improvement, and not easily addressable within physics courses themselves
  - Examples: algebraic operations
- Guidance to strengthen physics students' self-checking, "care-taking" skills offers greater hope for short-term improvements