The Development of Efforts to Address Physics Students' Mathematical Difficulties

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The Problem

- Difficulties with basic math skills impact performance of introductory physics students
- The difficulties are not resolved by students' previous mathematical training
- Students can't effectively grapple with physics ideas when they feel overburdened in dealing with calculational issues

Our Approach

- ***Assess*** nature and scope of difficulties using written diagnostic instruments and one-on-one oral interviews
- ***Address*** students' mathematical difficulties within the context of physics classes themselves, using in-class and out-of-class instructional materials

Overview: Requirements for Successful Application of Math to Physics

1. Understanding of mathematical **concepts**
2. Technical skill with mathematical **procedures**
3. Ability to apply in **physical context**
4. Ability to apply in **problem-solving context**

1. Understanding of Mathematical Concepts

- Recognition of meaning and significance of mathematical operations

Example [trigonometry]: Unknown sides and angles of a right triangle may be found by applying sine, cosine, and tangent functions to known sides and angles

Example [vectors]: Direction of a vector may be defined as the angle with respect to some fixed coordinate system

1. Understanding of Mathematical Concepts

- Sherin (2001): Students' understanding of the *concepts* underlying mathematical problem solving are central to success in physics
  - *Example [wave phenomena]:* Steinberg, Wittmann, and Redish (1997) probed students' understanding of mathematical concepts related to wave propagation
  - *Example [harmonic motion]:* Galle and Meredith (2014) developed tutorial worksheets to address students' confusion with meaning of, for example, $x(t) = 15\text{ cm} \cos(2\pi f t)$
- **How to address these problems:** Have students practice *explaining the meaning* of the mathematical expressions (Galle and Meredith, 2014)

1. Understanding of Mathematical Concepts

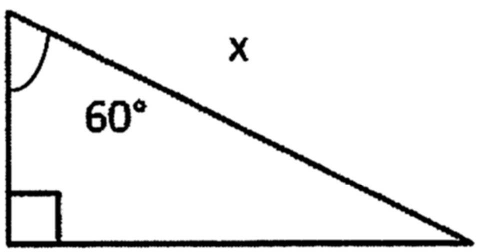
- **Trigonometry:** Many students are confused or unaware (or have forgotten) about the relationships between sides and angles in a right triangle (Meltzer and Jones, 2016).

- *Examples:* Questions from a diagnostic math test administered at Arizona State University, 2016-2017 (Administered as no-credit quiz during first week labs and/or recitation sections; calculators allowed)

Trigonometry Questions

with samples of correct student responses

1.



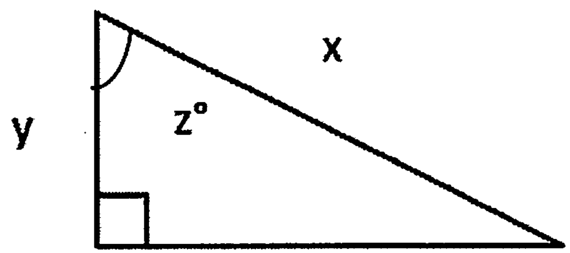
What is the value of x?

$$\cos 60 = \frac{10}{x}$$

$$x \cos 60 = 10$$

$$x = \frac{10}{\cos 60} = 20$$

2.



$\cos z = \frac{y}{x}$

What is the value of x?

A. ycos(z)

B. ycos(z)sin(z)

C. y/sin(z)

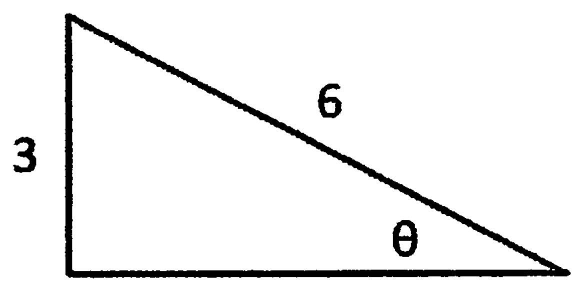
D. ysin(z)

E. ycos(z)/sin(z)

F. y/cos(z)

G. None of the above________________________________

3.



What is the value of θ?

$$\sin^{-1}(\theta) = \sin^{-1}\left(3/6\right)$$

$$\theta = 30^\circ$$

Trigonometry Questions:

Correct Response Rate (% *correct* responses), #1-3 combined

ASU Polytechnic campus, Spring + Fall average:

Algebra-based course, 1st semester, (*N* = 116): 37%

Algebra-based course, 2nd semester, (*N* =79): 48%

Results on Trigonometry Questions

Errors oberved: use of incorrect trigonometric function (e.g., cosine instead of sine), calculator set on radians instead of degrees, algebra errors; *unaware (or forgot) about inverse trigonometric functions, e.g., arctan, arcsin*, *arccos [*tan^{-1}, sin^{-1}, cos^{-1}*]*

– **How to address these problems:** It seems that students require substantial additional *practice and repetition* with basic trigonometric procedures

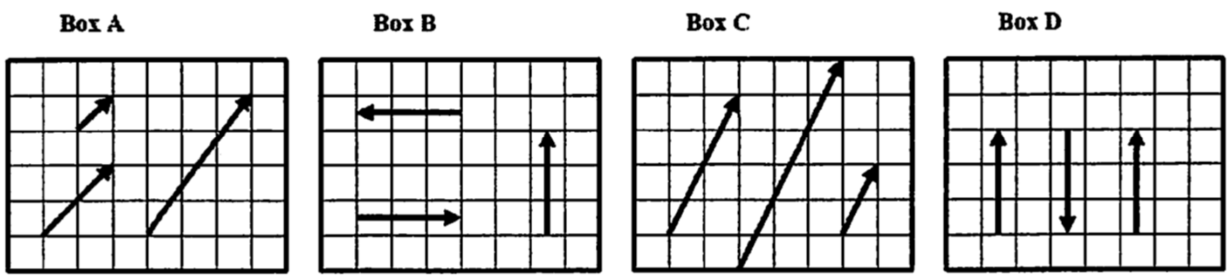
1. Understanding of Mathematical Concepts

- **Vector Concepts:** Many students are confused about the fundamental meaning of vector *direction*, as well as the role of direction in determining the resultant of two or more vectors (Nguyen and Meltzer, 2003; Barniol and Zavala, 2014; Meltzer and Jones, 2016).

Vector Direction Question

(Meltzer and Jones, 2016)

7. In the four boxes below are collections of vectors on top of equally spaced grid lines. Choose the answer from the list below that most correctly describes the comparative **directions** of the vectors within each box.



Possible answers. Select the best one.

A. Box **A** has all vectors with the same direction
B. Box **B** has all vectors with the same direction
C. Box **C** has all vectors with the same direction
D. Box **D** has all vectors with the same direction
E. Both boxes **A** and **C** have vectors that all have the same direction
F. Both boxes **A** and **D** have vectors that all have the same direction
G. Both boxes **C** and **D** have vectors that all have the same direction
H. The boxes, **A**, **C**, and **D** have vectors that all have the same direction
I. None of the above boxes have vectors with the same direction

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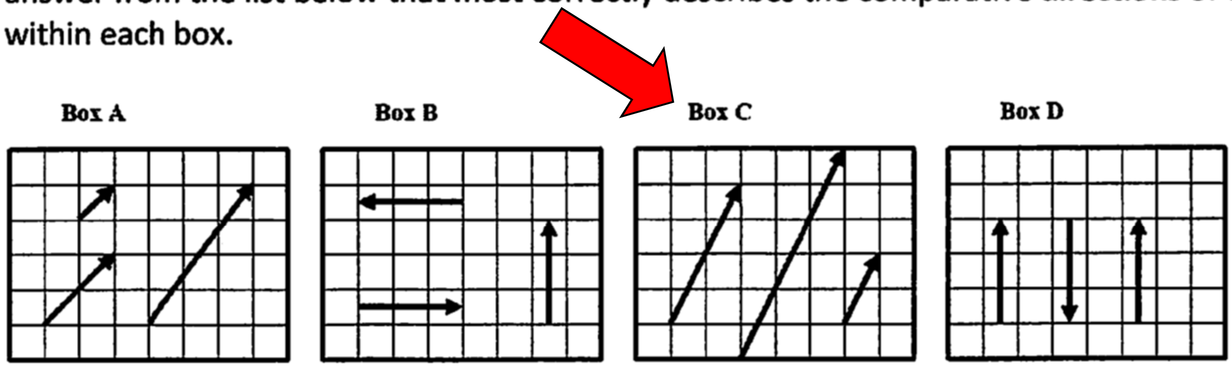

All same direction

Possible answers. Select the best one.

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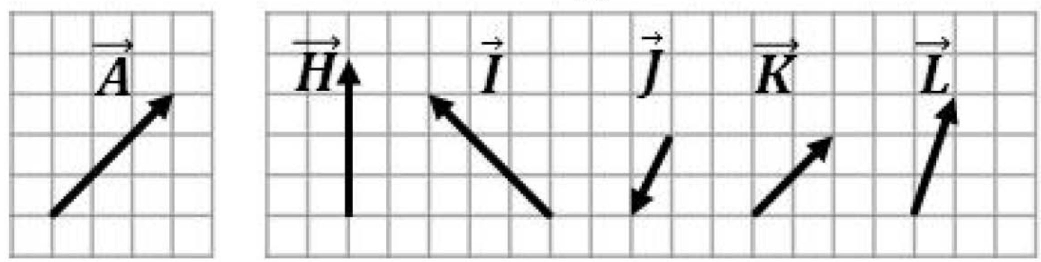
H. The boxes, **A**, **C**, and **D** have vectors that all have the same direction
I. None of the above boxes have vectors with the same direction

Vector Direction Question

(Barniol and Zavala, 2014)

5. The figure below shows vector $\vec{A}$ and a list of vectors. Which vector(s) has/have the same direction as vector $\vec{A}$?



(A) $\vec{K}, \vec{L}$
(B) $\vec{I}, \vec{K}$
(C) $\vec{K}$
(D) $\vec{H}, \vec{K}, \vec{L}$
(E) No vector has the same direction as vector $\vec{A}$

Vector Direction Question

(Barniol and Zavala, 2014)

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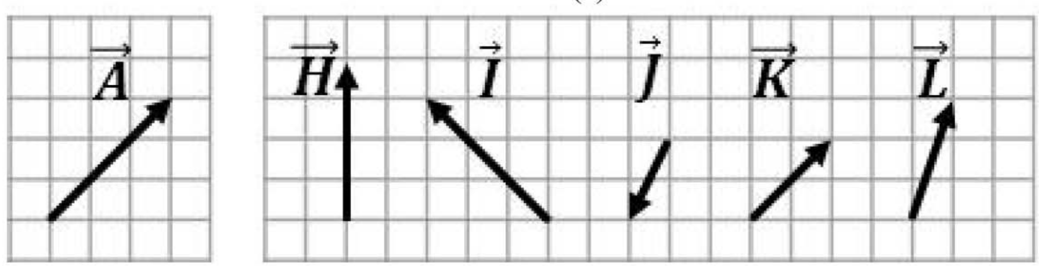


(A) $\vec{K}, \vec{L}$
(B) $\vec{I}, \vec{K}$
(C) $\vec{K}$
(D) $\vec{H}, \vec{K}, \vec{L}$
(E) No vector has the same direction as vector $\vec{A}$

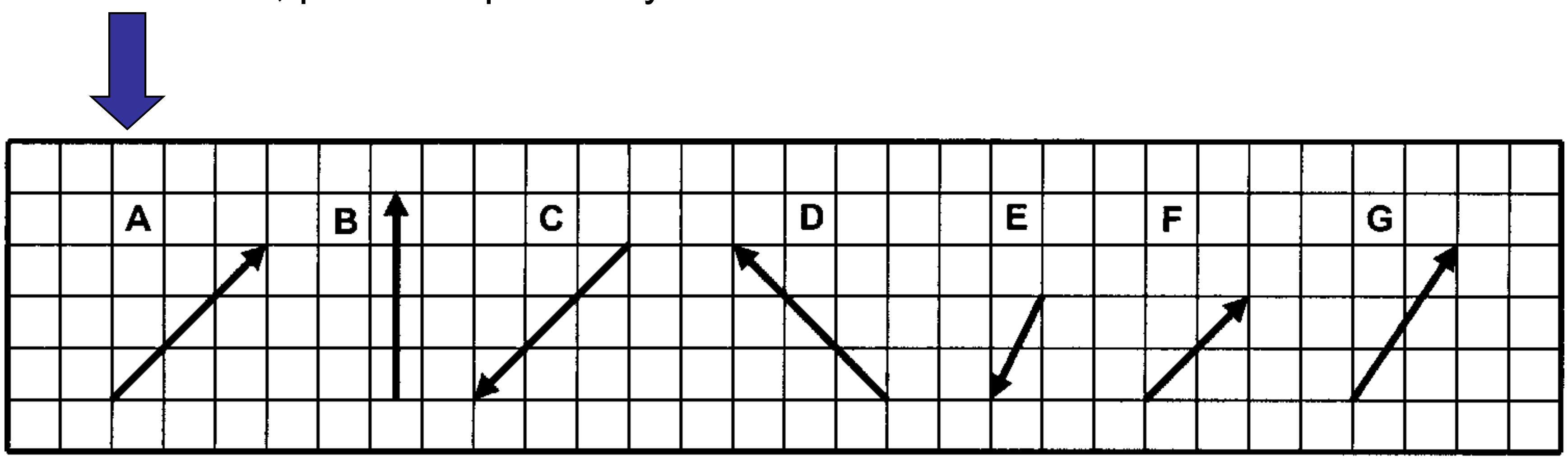
For comparison…

A very similar question administered to students at Iowa State University in 2003:

Vector Direction, ISU Diagnostic

(Nguyen and Meltzer, 2003)

2. List all the vectors that have the same **direction** as the first vector listed, $\vec{\mathbf{A}}$. If there are none, please explain why.



Explain ___________________________

Vector Direction, ISU Diagnostic

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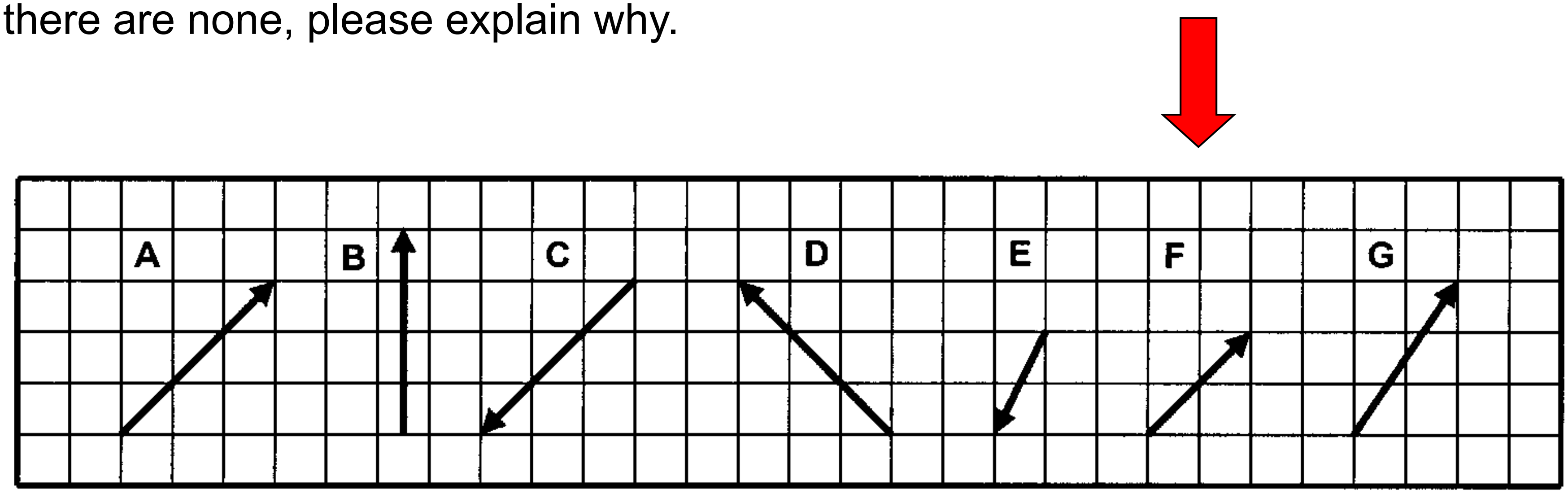

Explain ___________________________

Results on Vector Direction

Percent Correct Responses

Calculus-based physics, third semester

Tecnológico de Monterrey *[N = 423]* (2014): 86%

[Barniol and Zavala, 2014]

Calculus-based physics, second semester

Arizona State University *[N = 29]* (2016): 66%

Iowa State University *[N = 702]* (2003): 77%

Calculus-based physics, first semester

Arizona State University *[N = 104]* (2016): 51%

Iowa State University *[N = 608]* (2003): 71%

Results on Vector Direction

Percent Correct Responses

Algebra-based physics, second semester

Arizona State University *[N = 52]* (2016): 40%

Iowa State University *[N = 201]* (2003): 64%

Algebra-based physics, first semester

Arizona State University *[N = 72]* (2016): 40%

Iowa State University *[N = 520]* (2003): 55%

- Fewer correct responses in the algebra-based course

Vector Direction, Most Common Error

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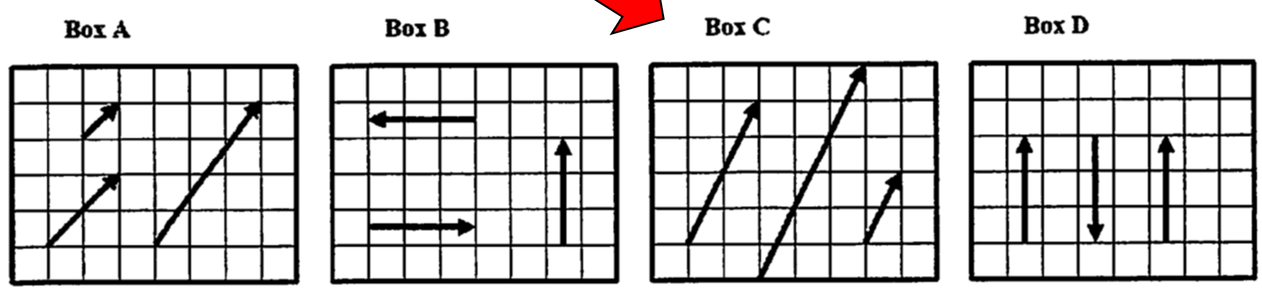

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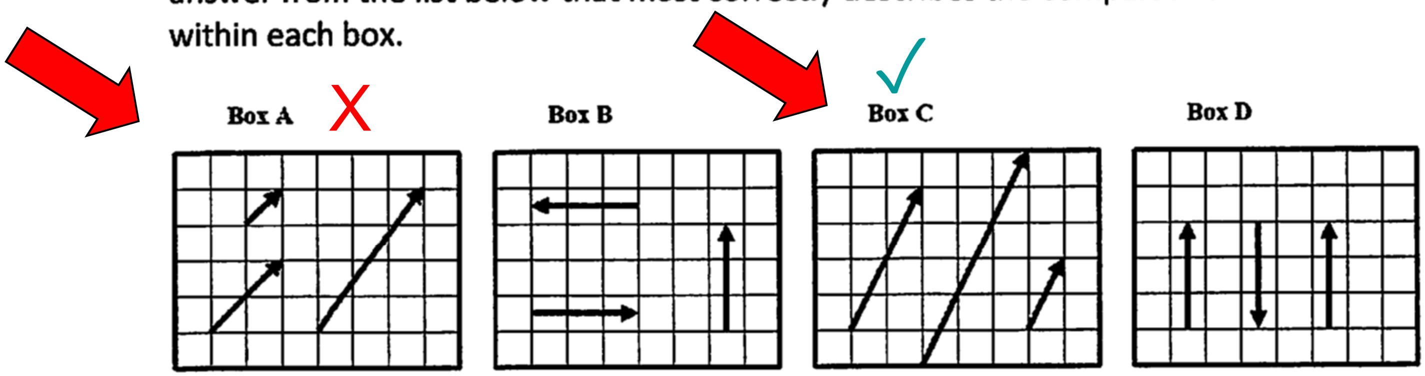

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1. Understanding of Mathematical Concepts

- **Vector Concepts:** Many students are confused about the fundamental meaning of vector *direction*, as well as the role of direction in determining the resultant of two or more vectors (Nguyen and Meltzer, 2003; Barniol and Zavala, 2014; Meltzer and Jones, 2016).
- **How to address these problems:** Students must first become aware of the "special" [very precise] meaning of "direction" in physics. They must practice problems to recognize that when adding or subtracting vectors, the *exact angle* of the vector with respect to some coordinate system must be preserved.

2. Technical Skill

- Ability to recognize meaning of specific symbols, and to execute specific mathematical procedures and operations
- *Example [trigonometry]:* Find hypotenuse of right triangle given one side and one angle
- *Example [vectors]:* Add two vectors represented in graphical form [i.e., as arrows in two dimensions]
- *Example [algebra]:* Given two equations with two unknown variables, solve to find value of the unknowns

2. Technical Skill: **Symbols**

- "**Language mismatches":** Students are confused by the very different symbols and techniques used in physics classes, for identical operations first seen in mathematics classes (Dray and Manogue, 1999-2004)
- **Unfamiliar symbols:** Students are often confused by new symbols or representations used in physics that are *not* used in mathematics classes, e.g., "arrow" representation of electric fields and gravitational forces; motion graphs (velocity-time, acceleration-time); "flux" [Φ] of electric field through a surface [Meltzer, 2005; Gire and Price, 2013]
- **Confusion of symbolic meaning:** Students perform worse on solving problems when symbols are used to represent common physical quantities in equations, e.g., "*m*" (mass [*masa*]) instead of "1.5 kg" [Torigoe and Gladding, 2007; 2011)

2. Technical Skill: **Symbols**

- "**Language mismatches":** Students are confused by the very different symbols and techniques used in physics classes, for identical operations first seen in mathematics classes (Dray and Manogue, 1999-2004)
  - *Example:* The "area element" used in vector calculus to do area integrals looks very different in physics textbooks, compared to mathematics textbooks

$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$ *[math, general expression]*

$dS = r^2 \sin\theta \, d\theta \, d\phi$ *[physics, for a sphere]*

2. Technical Skill: **Symbols**

- “**Language mismatches”:** Students are confused by the very different symbols and techniques used in physics classes, for identical operations first seen in mathematics classes (Dray and Manogue, 1999-2004)

- **How to address this problem (Dray and Manogue, 2004):**
  - Focus on “big ideas” that provide unification, instead of memorizing many formulas and procedures; e.g., infinitesimal line element on sphere, $d\mathbf{r} = dr\,\hat{r} + r \sin\theta\, d\theta\, \hat{\boldsymbol{\theta}} + r\, d\phi\, \hat{\boldsymbol{\phi}}$
  - Improve students’ geometric visualization skills, since physicists tend to think “geometrically” while math courses emphasize algebraic procedures. *Example:* manipulate vectors graphically as well as algebraically
  - Convey information using alternative means (e.g., diagrams, use of color) to help students generalize to three-dimensional physical world

2. Technical Skill: **Symbols**

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- **How to address this problem:**
  - Ensure that students have ample practice with representations used specifically in physics (e.g., diagrams, graphs, charts);
  - Include practice in "translating" between different representations (e.g., from "math" to "words" to "graphs")
  - Use "kinesthetic" activities to help students grasp geometrical meanings; Examples: "point fingers" in direction of vector gradient; use ruler and hoop to represent electrical flux (Gire and Price, 2012)

2. Technical Skill: **Symbols**

…udents are often confused by new …s that are *not* used in mathematics …presentation of electric fields and …tze…

…ble…

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2. Technical Skill: **Symbols**

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Example [University of Illinois]:

Version #1: A car can go from 0 to 60 m/s in 8 s. At what distance *d* from the start at rest is the car traveling 30 m/s? *[93% correct]*

Version #2: A car can go from 0 to v_1 in t_1 seconds. At what distance *d* from the start at rest is the car traveling (v_1/2)? *[57% correct]*

Much worse!

2. Technical Skill: **Symbols**

- **Confusion of symbolic meaning:** Students perform worse on solving problems when symbols are used to represent common physical quantities in equations, e.g., "*m*" (mass [*masa*] instead of "1.5 kg") [Torigoe and Gladding (2007; 2011)

- **How to address this problem (Torigoe and Gladding, 2011):**
  - In order to emphasize meaningful symbolic representation (instead of "plug and chug"), practice with problems that include or require:
    - compound expressions (such as $3m$ instead of m)
    - subscripts (such as m_A and m_B)
    - multiple or simultaneous equations (e.g., $a = \Delta v/t$ and $v^2=2ax$)

2. Technical Skill: **Procedures**

- **Vectors:** Students have difficulty interpreting and manipulating vector quantities represented as arrows [Nguyen and Meltzer, 2003; Barniol and Zavala, 2014)

Example: Add (or subtract) vectors A and B to find the resultant

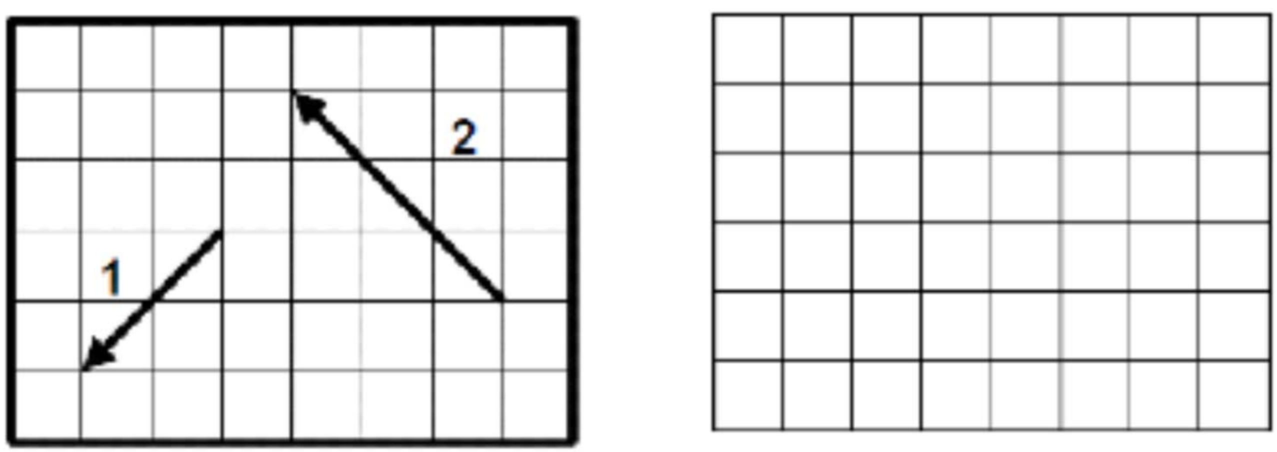
Addition of Vectors

(From Math Diagnostic Test, Arizona State University)

6)

In the figure below there are two vectors $\overline{\mathbf{1}}$ and $\overline{\mathbf{2}}$. In the empty grid, draw the sum or vector addition $\overline{\boldsymbol{R}}$ of the two (i.e., $\overline{\boldsymbol{R}} = \overline{\mathbf{1}} + \overline{\mathbf{2}}$).

Note: You can draw other vectors in the empty grid, but be sure to label $\overline{\boldsymbol{R}}$ clearly.



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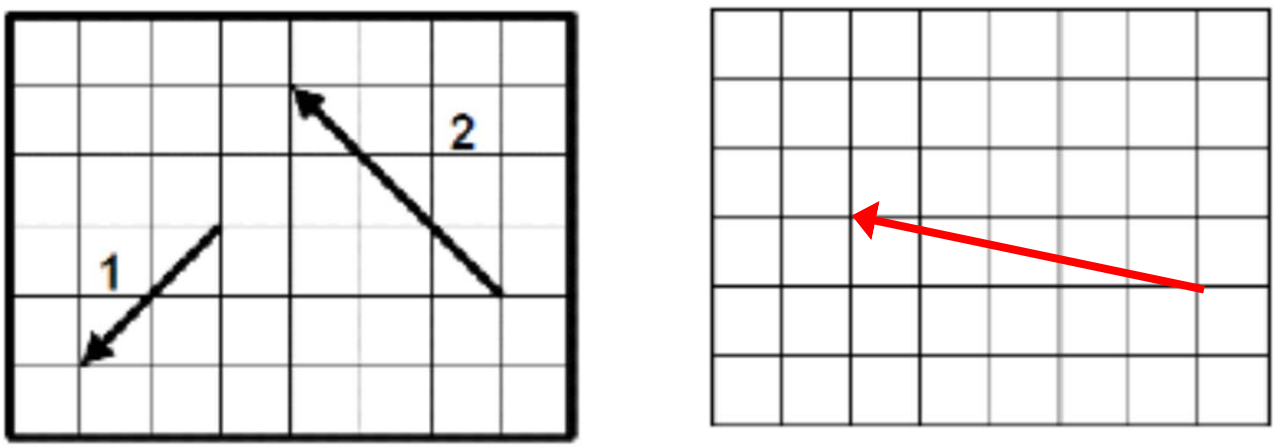

Addition of Vectors

(Algebra-based Course, 2nd semester, ASU)

Percent correct, Free-response Version (N = 61): 36%

6)

In the figure below there are two vectors $\overline{\mathbf{1}}$ and $\overline{\mathbf{2}}$. In the empty grid, draw the sum or vector addition $\overline{\boldsymbol{R}}$ of the two (i.e., $\overline{\boldsymbol{R}} = \overline{\mathbf{1}} + \overline{\mathbf{2}}$).

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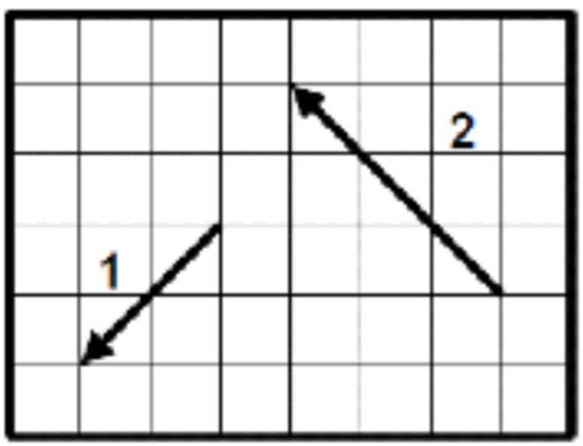
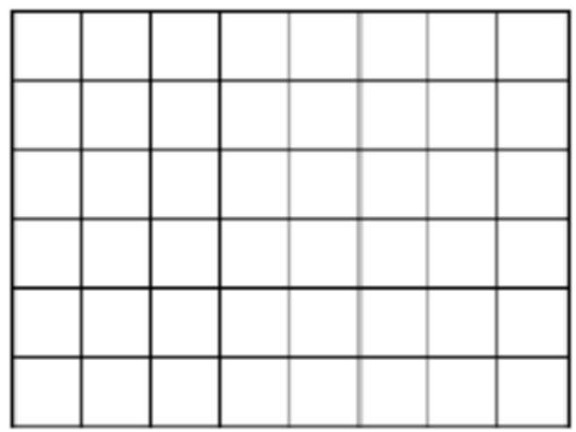
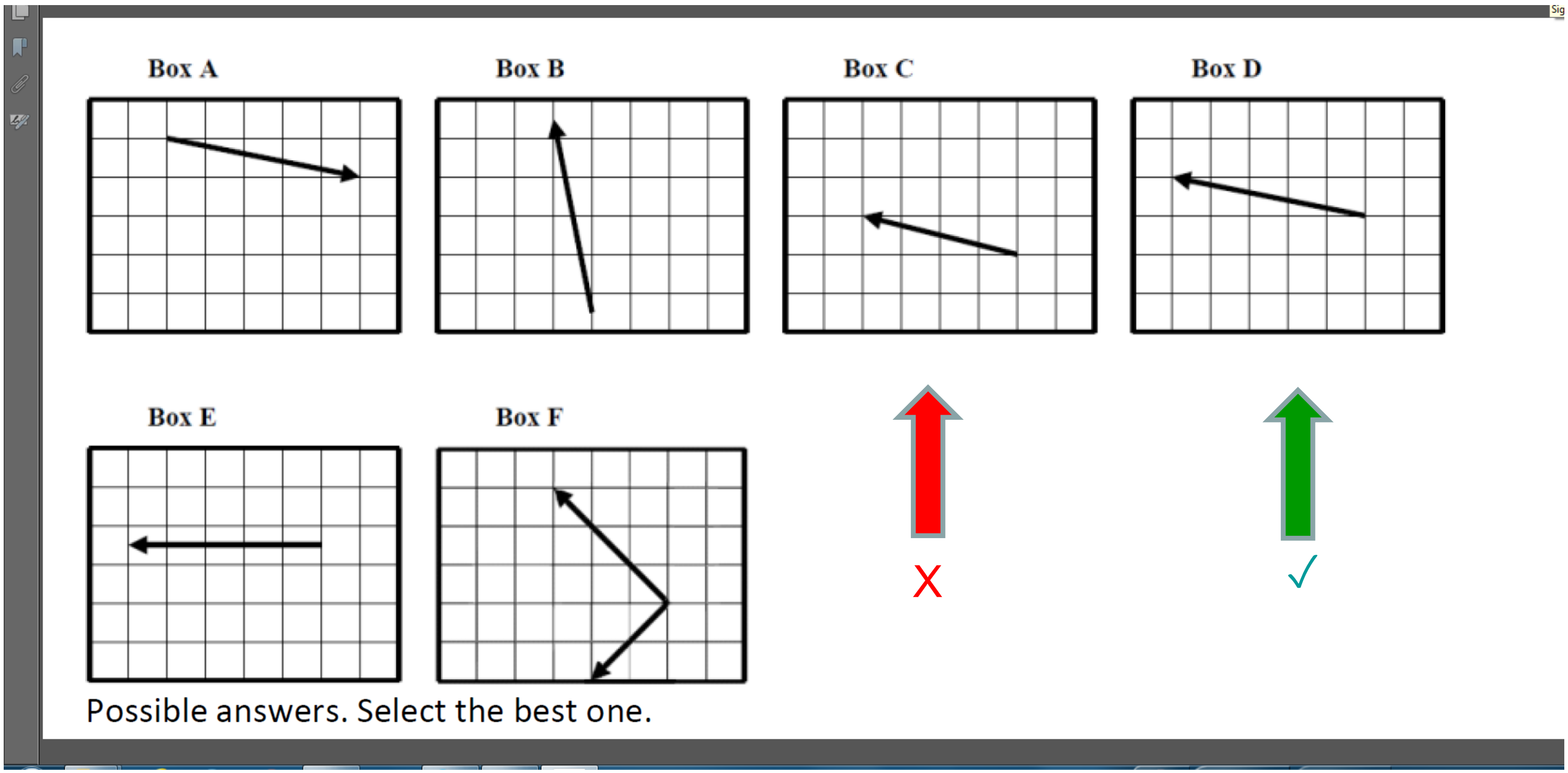




Addition of Vectors

(Algebra-based Course, 2nd semester, ASU)

Percent correct, Multiple-Choice Version (N = 62): 27%



Addition of Vectors

Percent Correct Responses

- Calculus-based course, first semester
  - Arizona State University: 71%
    - (multiple-choice: 68%)
  - Iowa State University: 58%
- Calculus-based course, second semester
  - Arizona State University: --
    - (similar problem, multiple-choice: 80%)
  - Iowa State University: 73%

Addition of Vectors

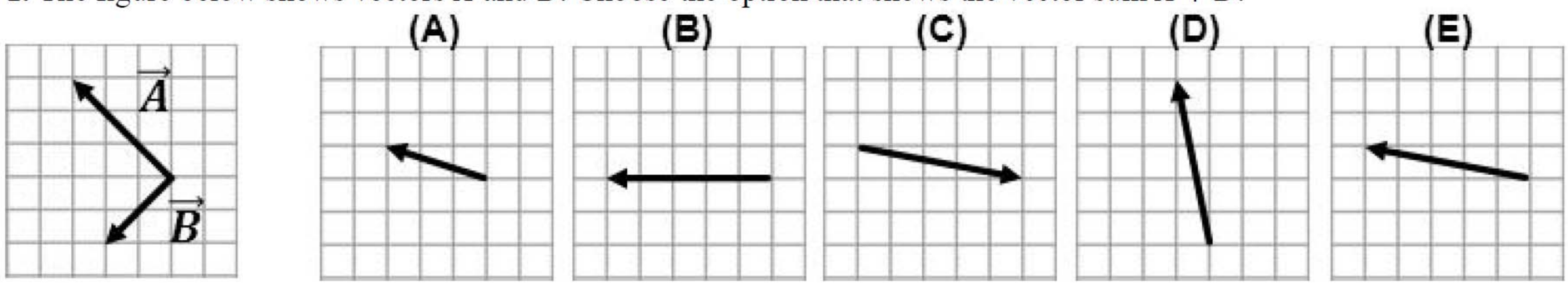
(Calculus-based Course, 3rd semester)

Tecnológico de Monterrey (Barniol and Zavala, 2014)

Percent correct, Multiple-choice, N = 423: 74%

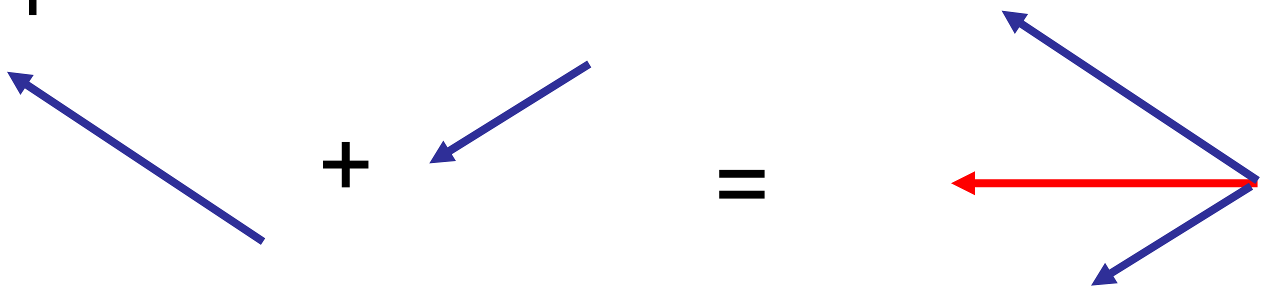
Test of Understanding of Vectors (TUV)

1. The figure below shows vectors $\vec{A}$ and $\vec{B}$. Choose the option that shows the vector sum $\vec{A} + \vec{B}$.

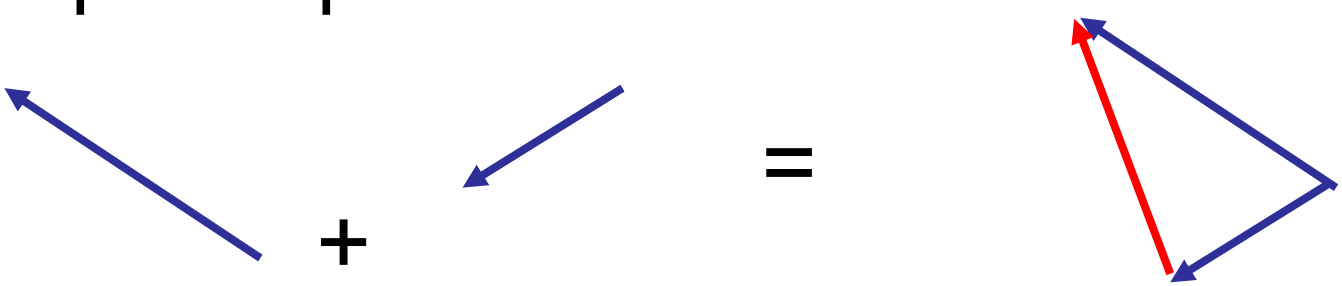


Common Student Errors With Vector Addition

- “Split the Difference” or “Bisector Vector”:



- “Tip-to-Tip”:



2. Technical Skill: **Procedures**

- **Vectors:** Students have difficulty interpreting and manipulating vector quantities represented as arrows [Nguyen and Meltzer, 2003; Barniol and Zavala, 2014)

How to address this problem:

  - Practice with a variety of vector orientations; introduce and use the "*ijk*" coordinate representation for vectors (Heckler and Scaife, 2015)
  - Design tutorial worksheet to aid students' understanding of scalar ("dot") product of vectors (Barniol and Zavala, 2016)
  - Provide extensive on-line practice and homework assignments related to frequently used vector procedures (Mikula and Heckler, 2017)

2. Technical Skill: **Procedures**

- **Algebra:** Students have difficulties in solving two simultaneous equations, and those difficulties are much greater when the equations are in symbolic form (Meltzer and Jones, 2016-2017)

Algebra: Simultaneous Equations

$3x = 2y$
$5x + y = 26$

What are the values of x and y? Show all your steps. For example, $x = 2$, $y = 5$ (These are NOT the correct answers).

Correct Response Rate, ASU (% correct responses)

Algebra-based course, second semester (*N* = 123): 70%

Algebra: Simultaneous Equations

$$x \cdot \cos(20°) = y \cdot \cos(70°)$$
$$x \cdot \cos(70°) + y \cdot \cos(20°) = 10$$

What are the values of x and y? Show all your steps. Note: The value for x should NOT include y, and the value for y should NOT include x.

Correct Response Rate, ASU (% correct responses)

Algebra-based course, second semester (N =150): 20-30%
(different campuses, slightly different versions)

Algebra: Simultaneous Equations

$$a \cdot \mathrm{x} = \mathrm{b} \cdot \mathrm{y}$$
$$b \cdot \mathrm{x} + \mathrm{a} \cdot \mathrm{y} = c$$

a, b, and c are constants.

What are the values of x and y in terms of a, b, and c? Show all your steps. Note: The value for x should NOT include y, and the value for y should NOT include x.

Algebra: Simultaneous Equations

$a \cdot x = b \cdot y$
$b \cdot x + a \cdot y = c$

a, b, and c are constants.
What are the values of x and y in terms of a, b, and c? Show all your steps. Note: The value for x should NOT include y, and the value for y should NOT include x.

$$x = \frac{by}{a}$$

$$b\left(\frac{by}{a}\right) + ay = c$$

$$\frac{b^2 y}{a} + ay = c$$

$$y\left(\frac{b^2}{a} + a\right) = c$$

$$y = \frac{c}{\left(\frac{b^2}{a} + a\right)}$$

$$x = \frac{b\left(\frac{c}{(b^2/a + a)}\right)}{a}$$

Sample of Correct Student Response

Algebra: Simultaneous Equations

$$a \cdot \mathrm{x} = \mathrm{b} \cdot \mathrm{y}$$
$$b \cdot \mathrm{x} + \mathrm{a} \cdot \mathrm{y} = c$$

a, b, and c are constants.
What are the values of x and y in terms of a, b, and c? Show all your steps. Note: The value for x should NOT include y, and the value for y should NOT include x.

Correct Response Rate, ASU (% correct responses)

Algebra-based course, second semester (*N* =150): 10-20%
(different campuses, slightly different versions)

Only 10-20% correct responses!

Sources of Difficulties

- Carelessness
  - Students *very frequently* self-correct errors during interviews
  - Evidence of carelessness on written diagnostic
- Skill practice deficit: Insufficient repetitive practice with learned skills
  - E.g., applying definitions of sine and cosine
- Conceptual confusion
  - E.g., not realizing that sides and angles of right triangle are related by trigonometric functions

How to Address Difficulties?

- Carelessness:
  - (1) review and check steps
  - (2) find alternative solutions
  - (3) habitual use of estimation
  - (4) apply dimensional analysis (for physical problems)
- Skill deficit: Practice and repetition
- Conceptual confusion: Review and study of basic ideas

3. Ability to Apply Mathematics in a Physical Context

- Student difficulties that *appear* to be mathematical in origin may actually be due in part to application in a *physical* context [Shaffer and McDermott, 2005; Christensen and Thompson, 2010-12; Thompson, Manogue, Roundy, and Mountcastle, 2012; Barniol and Zavala, 2010; Zavala and Barniol, 2013]

- *Example [calculus]:* Finding and comparing the "area under the curve" by applying the definite integral may be more challenging in a thermodynamics context (thermodynamic process represented on a Pressure-Volume diagram) [Christensen and Thompson, 2010-2012]

- *Example [vectors]:* The method used *and* the errors made by students when adding or subtracting vectors depend strongly on the specific physical context, and on whether there *is* a physical context [Shaffer and McDermott, 2005; Van Deventer and Wittmann, 2005; Barniol and Zavala, 2010]

3. Ability to Apply Mathematics in a Physical Context

- Student difficulties that *appear* to be mathematical in origin may actually be due in part to application in a *physical* context

How to address this problem:

- Mathematics procedures must be practiced in a variety of physical contexts, and students must be made aware of possible confusion introduced by the context

3. Ability to Apply Mathematics in a Problem-Solving Context

- Students often fail to make use of specific mathematical tools that they *do* know how to use, because they don't recognize their applicability to a physics problem [Bing and Redish, 2009; Gupta and Elby, 2011]

How to address this problem:

- – Ask open-ended questions, and vary the physical context, to provide a wide range of possible responses
- – Become aware of how students interpret problems; "exaggerate" cues regarding appropriate solution pathways (Bing and Redish, 2009)
- – Guide students to seek coherence between formal and "everyday" reasoning (Gupta and Elby, 2011)

Summary:

What Options do Physics Instructors Have for Dealing with Students' Mathematics Difficulties?

- Test to assess scope of problem
- Take time to review basic math
- Assign or suggest out-of-class math review practice
  [We will be developing appropriate instructional materials]
- Reduce mathematical burden of syllabus
  - more qualitative problems, fewer problems requiring algebraic manipulation
- Nothing: Leave it up to the students